Commutative Algebra III

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Contents

This course will cover a selection of basic topics in commutative algebra. I will be assuming knowledge of a first course in commutative algebra, as in the book of Atiyah-MacDonald [1]. I will also assume knowledge of Tor and Ext. Some of topics which will covered may include Cohen-Macaulay rings, Gorenstein rings, regular rings, Gröbner bases, the module of differentials, class groups, Hilbert functions, Grothendieck groups, projective modules, tight closure, and basic element theory (see [4]). Eisenbuds book [3], the book of Bruns and Herzog [2], and Matsumuras book [5] are all good reference books for the course, but there is no book required for the course.

Chapter 1

Hilbert Functions and Multiplicities

Through out these notes, we will need the concept of a graded object. A graded ring $R = \bigoplus_{i \geqslant 0} R_i$ is a commutative ring with identity, decomposed as a direct sum of abelian groups with

$$R_i \cdot R_j \subseteq R_{i+j}.$$

In particular, each R_i is an R_0-module and R_0 is a commutative ring itself ($1 \in R_0$). Likewise, a graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is an R-module, decomposed as a direct sum of abelian groups with

$$R_i \cdot M_j \subseteq M_{i+j}.$$

Each M_n is an R_0-module.

If R is a graded ring and M, N are graded R-modules, then an R-homomorphism $f : M \to N$ is said to be homogeneous of degree k if $f(M_n) \subseteq M_{n+k}$ for all n. Homogeneous maps are very desirable, so we define a convention to transform graded maps in to homogeneous maps. By twist, denoted $M(n)$, we mean a new graded module (the same as M with out grading), but

$$M(n) := M_{i+n}.$$

Example 1. Let R be a graded ring. The new ring $R(n)$ is a graded free module, isomorphic to R, but has a generator in degree $-n$. So, $R(-n)$ has a generator in degree n.

1 Hilbert Functions

In the following, we denote the length of an R-module M by $\lambda_R(M)$. For more information on length, see [1].

Definition. Let R is a graded ring and M a graded R-module with finite length. The Hilbert series of M is

$$H_M(t) := \sum_{i\in\mathbb{Z}} \lambda_{R_0}(M_i)t^i.$$

1.1 Examples

Before we begin an in-depth study of the Hilbert series, we consider a few examples.

Example 2. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k with $\deg(x_i) = 1$ for all $i = 1 \ldots d$. Here we have that

$$\begin{aligned} R_n &= k\langle \text{monomials of degree } n\rangle \\ &= k\langle x_1^{a_1} \ldots x_d^{a_d} \mid \sum a_i = n\rangle. \end{aligned}$$

It is a standard fact that the length of a vector space over a field is the same as the vector space dimension. Thus, $\dim_k(R_n) = \binom{n+d-1}{d-1}$ and the Hilbert series is given by

$$H_R(t) = \sum_{i=0}^{\infty} \binom{n+d-1}{d-1} t^i.$$

Example 3. Let $k[x, y]$ be a polynomial ring over a field k and define

$$R = k[x^l, x^{l-1}y, \ldots, xy^{l-1}, y^l] \subseteq k[x, y].$$

Assume that each element of $\{x^l, x^{l-1}y, \ldots, xy^{l-1}, y^l\}$ has degree one. Notice that

$$\dim_k(R_i) = \binom{il+2-1}{1} = il + 1,$$

which gives us a Hilbert series of

$$H_R(t) = \sum_{i=0}^{\infty} (il + 1)t^i.$$

Further, we are able to write $H_R(t)$ as a rational function by using differentiation:

$$\begin{aligned} H_R(t) &= \sum_{i=0}^{\infty} l(i+1)t^i - (l-1)\sum_{i=0}^{\infty} t^i \\ &= l\left(\sum t^i\right)' - (l-1)\sum t^i \\ &= l\left(\frac{1}{1-t}\right)' - \frac{l-1}{1-t} \\ &= \frac{l}{(1-t)^2} - \frac{l-1}{1-t} \\ &= \frac{(l-1)t+1}{(1-t)^2} \end{aligned}$$

Example 4. Let R be a hypersurface of degree m. That is, let $f \in S_m$ where $S = k[x_1, \ldots, x_d]$ and $\deg x_i = 1$. Now Set $R = S/(f)$. To find $H_R(t)$, consider the short exact sequence

$$0 \longrightarrow S(-m) \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0.$$

Notice that multiplication by f is a degree zero homomorphism. Since length is additive, the Hilbert series is

$$\begin{aligned} H_R(t) &= H_S(t) - H_{S(-m)}(t) \\ &= \frac{1}{(1-t)^d} - \frac{t^m}{(1-t)^d}. \end{aligned}$$

To see the Hilbert series of $S(-m)$, notice that

$$\begin{aligned} H_{S(-m)}(t) &= \sum_{i\in\mathbb{Z}} \dim_k S(-m)_i t^i \\ &= \sum_{i\in\mathbb{Z}} \dim_k S_{i-m} t^i \\ &= \sum_{j=i-m\in\mathbb{Z}} \dim_k S_j t^{j+m} \\ &= t^m H_S(t) \end{aligned}$$

Remark 1. Suppose that $S = k[x_1, \ldots, x_n]$ with the usual grading. Let $f_1, \ldots, f_r$ be a regular sequence with $\deg(f_i) = d_i$. It is natural to guess that

$$H_{S/(f_1,\ldots,f_r)}(t) = \frac{\prod(1-t^{d_i})}{(1-t)^n}.$$

Notice that if $r = n$, then $S/(f_1, \ldots, f_n)$ has finite length. Then $H_{S/(f_1,\ldots,f_n)}(t)$ is a polynomial! (Total length being the product of the degrees of f_i.) Trying the guess we find that

$$H_R(t) = \frac{\prod(1-t^{d_i})}{(1-t)^n} = \prod_{i=1}^{n}(1 + t + \cdots + t^{d_i-1}).$$

So, $H_R(1) = d_1 d_2 \cdots d_n$.

Example 5. Let $R = k[x, y]$ be a polynomial ring over a field k in two variables. Let $\deg(x) = 2$ and $\deg(y) = 3$. Calculating the length of each graded piece gives us the following:

$i =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\dim_k(R_i) =$	1	0	1	1	1	1	2	1	2	2	2	2	3	2

Given the short exact sequence

$$0 \longrightarrow R(-2) \xrightarrow{\cdot x} R \longrightarrow k[y] \longrightarrow 0,$$

we have that $H_R(t) = t^2 H_R(t) + H_{k[y]}(t)$. Since $H_{k[y]} = 1/(1-t^3)$, we have

$$H_R(t) = \frac{1}{(1-t^2)(1-t^3)}.$$

Remark 2. Consider $R = k[x,y]/J$ where J is the set of all differences of polynomials of the same degree. Notice that $\dim_k(R_i) = 1$ for all $i \geqslant 0$. It can be shown that

$$R = \frac{k[x,y]}{(x^3 - y^2)} \simeq k[t^2, t^3].$$

Example 6. Another type of grading is multi-degree. For an example of this, let $R = k[x,y]$ be a polynomial ring in two variables. We let the variables have the following degrees in $\mathbb{N}^2$:

$$\deg(x) = (1,0)$$
$$\deg(y) = (0,1).$$

Here we have the $R_{(i,j)} = k\langle x^i y^j \rangle$. In this case,

$$H_R(t,s) = 1 + t + s + t^2 + st + s^2 + \cdots$$

1.2 The Hilbert-Samuel Polynomial

Before we define the Hilbert-Samuel polynomial, we need some propositions.

Proposition 1. *Let R be a graded ring. The following are equivalent:*

(1) R is noetherian;

(2) R_0 is noetherian and $R_+ = \oplus_{i \geqslant 1} R_i$ is a finitely generated ideal;

(3) R_0 is noetherian and $R \simeq R_0[x_1, \ldots, x_n]/I$ with $\deg(x_i) = k_i$ where I is a homogeneous ideal.

Proof. (3) $\Rightarrow$ (1): This follows from the Hilbert basis theorem.

(1) $\Rightarrow$ (2): The object R_+ is an ideal and hence is finitely generated as R is noetherian. We thus have that $R_0 \simeq R/R_+$ is noetherian as well.

(2) $\Rightarrow$ (3): Choose $z_1, \ldots, z_n \in R_+$ with $\deg(z_i) = k_i$ such that $R_+ = Rz_1 + \cdots + Rz_n$. We claim that $R = R_0[z_1, \ldots, z_n]$. As we naturally have that $R \supseteq R_0[z_1, \ldots, z_n]$, it is enough to show $R_i \subseteq R_0[z_1, \ldots, z_n]$ for all i. To do this we induct on i. The $i = 0$ is clear. Let $i > 0$ and suppose the claim is true up to $i - 1$. Let $f \in R_+$ and write it as

$$f = \sum_{j=1}^{n} s_j z_J$$

where $s_j \in R$. If we restrict to the degree i part,

$$f = \sum_{j=1}^{n} s'_j z_J$$

where $s'_j \in R_{i-k_j}$. By induction we are done because $s'_j \in R_0[z_1, \ldots, z_n]$. $\square$

Remark 3. If M is an R-module, then M is naturally an $R/\mathrm{ann}(M)$-module.

Proposition 2. *Let R be a noetherian graded ring and M a finitely generated graded R-module. Then for all $n \in \mathbb{Z}$, M_n is a finitely generated R_0-module. In particular, if R_0 is artinian, then the length of M_n is finite.*

Proof. Consider the submodule of M defined by $M_{\geqslant n} := \oplus_{i \geqslant n} M_i$. Since M is noetherian, $M_{\geqslant n}$ is a finitely generated graded R-module. Therefore we have that $M_n := M_{\geqslant n}/M_{\geqslant n+1}$ is also a finitely generated graded R-module. By Remark 3, we have that M_n is a finitely generated graded $R/\mathrm{ann}(M_n)$-module as well. But $R_+ = \mathrm{ann}(M_n)$, hence M_n is a finitely generated $R_0 = R/R_+$-module. $\square$

Theorem 3. *Let R be a graded noetherian ring with R_0 artinian and let M be a finitely generated graded R-module. Write $R = R_0[z_1, \dots, z_s]$ with $\deg(z_i) = k_i$. Then*

$$H_M(t) = \frac{f_M(t)}{\prod_{i=1}^{s}(1 - t^{k_i})}$$

where $f_M(t) \in \mathbb{Z}[t, t^{-1}]$. If M is non-negatively graded, then we have that $f_M(t) \in \mathbb{Z}[t]$.

Proof. By Proposition 2, the length of M_n is finite for all n. Induct on s. For the $s = 0$ case, let $R = R_0$ and assume that M is finitely generated over R_0. Thus $M = \oplus_{i=-p}^{r} M_i$ with $\lambda(M_i) < \infty$ and

$$H_M(t) = \sum \lambda(M_i) t^i \in \mathbb{Z}[t, t^{-1}].$$

Assume that $s > 0$ and consider the following exact sequence:

$$0 \longrightarrow K \longrightarrow M(-k_s) \xrightarrow{\cdot z_s} M \longrightarrow C \longrightarrow 0$$

where K and C are the kernel and cokernel of the map defined by multiplication by z_s. Both K and C are finitely generated R-modules. Note that $z_s K = z_s C = 0$. Therefore K and C are modules over $R_0[z_1', \dots, z_{s-1}'] = R/z_s R$. We have that

$$\begin{aligned} H_M(t) + H_K(t) &= H_{M(-k_s)}(t) + H_C(t) \\ &= t^{k_s} H_M(t) + H_C(t). \end{aligned}$$

Hence, we can solve for $H_M(t)$ to get

$$\begin{aligned} H_M(t) &= \frac{H_C(t) - H_K(t)}{(1 - t^{k_s})} \\ &= \frac{f_c(t) - f_k(t)}{\prod_{i=1}^{s}(1 - t^{k_i})}. \end{aligned}$$

If M is non-negatively graded, then C is non-negatively graded as well. Thus $f_C(t)$ is in $\mathbb{Z}[t]$. Also, $M(-k_s)$ is non-negatively graded since $k_s > 0$. Therefore K is also non-negatively graded, thus $f_K(t)$ is also in $\mathbb{Z}[t]$. $\square$

Corollary 4. *Suppose that in Theorem 3 we have $k_i = 1$ for $i = 1, \ldots, s$ and that M is non-negatively graded. Then there exists a polynomial, $P_M(x)$ in $\mathbb{Q}[x]$ such that $\lambda(M_n) = P_M(n)$ for all $n >> 0$. Moreover, $\deg(P_M) \leqslant s-1$.*

Definition. The polynomial $P_M(x)$ is called the Hilbert-Samuel polynomial.

Proof. Note that we have for some $f(t)$ in $\mathbb{Z}[t]$,

$$\sum_{i \geqslant 0} \lambda(M_i) t^i = \frac{f(t)}{(1-t)^s} \tag{1.1}$$

$$= f(t) \sum_{i \geqslant 0} \binom{s+i-1}{s-1} t^i \tag{1.2}$$

If $\deg(f) = N$, we can write

$$f(t) = a_N t^N + \cdots + a_o$$

where $a_i \in \mathbb{Z}$. The coefficient of t^n is a polynomial in n of degree $s-1$ with coefficients in $\mathbb{Q}$. In particular,

$$\lambda(M_n) = \sum_{j=0}^{N} \binom{s+n-j-1}{s-1}.$$

Here we have that

$$\binom{s+n-j-1}{s-1} = \frac{\overbrace{(s+n-j-1)(s+n-j-2)\cdots(n-j+1)}^{s-1}}{(s-1)!}$$
$$= \frac{n^{s-1}}{(s-1)!} + \text{ lower terms.}$$

We now set

$$P_M(x) = \sum_{j=0}^{N} a_j \binom{s+x-j-1}{s-1}.$$

□

Example 7. Let $R = k[x,y]$ and assume that $\deg(x) = 2$ and $\deg(y) = 3$. Here we have that

$$H_M(t) = \frac{1}{(1-t^2)(1-t^3)} = (1 + t^2 + t^4 + \cdots)(1 + t^3 + t^6 + \cdots)$$

Thus the length is given by the coefficients of t^n, that is,

$$\lambda(R_n) = |\{(a,b) \mid 2a + 3b = n\}|$$

and this is not a polynomial but a quasi-polynomial.

Fact. Recall that
$$\binom{x+i}{i} = \frac{(x+i)(x+i-1)\cdots(x+1)}{i!}.$$
Then $\{\binom{x+i}{i}\}_{i=0}^{\infty}$ are a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$.

Example 8. Notice that
$$x^2 = 2\binom{x}{2} + \binom{x}{1}.$$
We can use this to find the sum of the first n squares. That is,
$$\begin{aligned}
1 + 2^2 + 3^2 + 4^2 + \cdots + n^2 &= \sum_{x=1}^{n} x^2 \\
&= 2\sum_{x=1}^{n}\binom{x}{2} + \sum_{x=1}^{n}\binom{x}{1} \\
&= 2\binom{n+1}{3} + \binom{n+1}{1} \\
&= \frac{2n(n-1)(n-2)}{6} + \frac{3n(n-1)}{6} \\
&= \frac{(2n+1)(n(n+1))}{6}.
\end{aligned}$$
Remark 4. We have $\sum_{j=0}^{n}\binom{j+k}{k} = \binom{n+k+1}{k+1}$.

Remark 5. Notice that $\left\{\binom{x+i}{i} : i \geq 0\right\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$.

Remark 6. If $f(x) \in \mathbb{Q}[x]$ and we write:
$$f(x) = \sum_{j=0}^{n} b_j \binom{x+j}{j}$$
and and we assume $b_n \neq 0$, then $f(x)$ is a polynomial of degree n with leading coefficient $\frac{b_n}{n!}$. Moreover, if we set:
$$g(s) := \sum_{i=0}^{s} f(i),$$
then $g(s)$ is a polynomial of degree $n+1$ with leading coefficient $\frac{b_n}{(n+1)!}$.

Proof. Clearly $\deg f = n$ and the leading coefficient is $\frac{b_n}{n!}$. Consider now
$$g(s) = \sum_{i=0}^{s} f(i) = \sum_{i=0}^{s}\sum_{j=0}^{n} b_j \binom{i+j}{j} = \sum_{j=0}^{n} b_j \sum_{i=0}^{s}\binom{i+j}{j} = \sum_{j=0}^{n} b_j \binom{s+j+1}{j+1},$$
which is now a polynomial of degree $n+1$ and leading coefficient $\frac{b_n}{(n+1)!}$. □

Remark 7. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree n. Write $f(x) = \sum_{j=0}^{n} b_j\binom{x+j}{j}$. If $f(m) \in \mathbb{N}$ for all $m >> 0$, then $b_j \in \mathbb{Z}$ for all j and $b_n > 0$.

2 Multiplicities

Throughout this section let $(R, \mathfrak{m}, k)$ be a noetherian local ring, where $\mathfrak{m}$ denotes the unique maximal ideal and $k = R/\mathfrak{m}$. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal (i.e. $\sqrt{I} = \mathfrak{m}$) and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$. Define

$$\mathrm{gr}_I R =: G = \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}},$$

where $I^0 = R$, which is called the associated graded ring of R with respect to I. It is a ring with the product

$$\begin{gathered} I^n/I^{n+1} \times I^m/I^{m+1} \to I^{n+m}/I^{n+m+1} \\ (r^*, s^*) \mapsto r^* s^* \end{gathered}$$

on the graded components.

Remark 8. $G = G_0[G_1]$ is a noetherian graded ring. This is because $G_0 = R/I$ is artinian, hence noetherian. Also R is noetherian, therefore I is finitely generated and so the ideal $G_+ = \oplus_{n \geq 1} I^n/I^{n+1} = G \cdot G_1$ is finitely generated too. We conclude by Proposition 1.

Definition. Let $I \subseteq R$ and M be as above. Define

$$\mathfrak{M}(I) := \bigoplus_{n \geq 0} \frac{I^n M}{I^{n+1} M}.$$

This is a graded G-module generated in degree zero, so it is finitely generated:

$$\mathfrak{M}(I) = G\left(\mathfrak{M}(I)_0\right).$$

Corollary 5. *$\lambda\left(I^n M/I^{n+1}M\right) = Q(n)$ is a polynomial in n (for $n >> 0$) of degree at most $\mu(I) - 1$, where $\mu(I) = \lambda(I/\mathfrak{m}I)$ is the minimal number of generators of the ideal I.*

Proof. Since G is generated in degree one, by Corollary 4 the degree of the polynomial is at most $\mu(G_1) - 1$. But

$$\mu(G_1) = \mu(I/I^2) = \mu(I),$$

where the last equality follows by NAK (Nakayama's Lemma). □

Corollary 6. *With the same assumptions as above we have that $\lambda(M/I^n M)$ is a polynomial in n, for $n >> 0$, having positive leading coefficient and degree bounded by $\mu(I)$.*

Proof. Notice that

$$\lambda(M/I^n M) = \sum_{j=0}^{n-1} \lambda\left(\frac{I^j M}{I^{j+1} M}\right).$$

Then we conclude by Remark 7 and Corollary 5. □

Definition. We set $P_{I,M}(n)$ to be the polynomial such that

$$P_{I,M}(n) = \lambda\left(\frac{M}{I^n M}\right) \quad \text{for } n >> 0$$

and we call it the Hilbert polynomial of M with respect to I.

Theorem 7. $\deg P_{I,M} = \dim M$.

Before proving it we need some more results.

Lemma 8. *Let the notation be as above and assume*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

is a short exact sequence of finitely generated R-modules. Then

(1) $\deg P_{I,M} = \max\{\deg P_{I,N}, \deg P_{I,L}\}$.

(2) $\deg(P_{I,M} - P_{I,N} - P_{I,L}) < \deg P_{I,M}$.

Proof. (1) Tensor the short exact sequence with R/I^n, then we get:

$$\frac{N}{I^n N} \longrightarrow \frac{M}{I^n M} \longrightarrow \frac{L}{I^n L} \longrightarrow 0$$

and hence

$$\lambda\left(\frac{M}{I^n M}\right) \leq \lambda\left(\frac{N}{I^n N}\right) + \lambda\left(\frac{L}{I^n L}\right).$$

This implies (for $n >> 0$):

$$\deg P_{I,M} = \deg\left(\lambda\left(\frac{M}{I^n M}\right)\right) \leq \deg\left(\lambda\left(\frac{N}{I^n N}\right) + \lambda\left(\frac{L}{I^n L}\right)\right) =$$

$$= \max\left\{\deg\left(\lambda\left(\frac{N}{I^n N}\right)\right), \deg\left(\lambda\left(\frac{L}{I^n L}\right)\right)\right\} = \max\{\deg P_{I,N}, \deg P_{I,L}\}.$$

Conversely there exists a short exact sequence:

$$0 \longrightarrow \frac{N}{I^n M \cap N} \longrightarrow \frac{M}{I^n M} \longrightarrow \frac{L}{I^n L} \longrightarrow 0$$

and also, by Artin-Rees Theorem, there exists $k \in \mathbb{N}$ such that for all $n \geq k$

$$I^n M \cap N = I^{n-k}(I^k M \cap N) \subseteq I^{n-k} N.$$

This implies

$$\lambda\left(\frac{N}{I^n M \cap N}\right) \geq \lambda\left(\frac{N}{I^{n-k} N}\right)$$

for $n >> 0$. Therefore, for such n:

$$\lambda\left(\frac{M}{I^nM}\right) = \lambda\left(\frac{L}{I^nL}\right) + \lambda\left(\frac{N}{I^nM \cap N}\right) \geq \lambda\left(\frac{L}{I^nL}\right) + \lambda\left(\frac{N}{I^{n-k}N}\right).$$

But $P_{I,N}(n-k)$ and $P_{I,N}(n)$ have same degree and leading coefficient (the first is just a translation of the second one). Hence

$$\deg P_{I,M} = \deg\left(\lambda\left(\frac{M}{I^nM}\right)\right) \geq \deg\left(\lambda\left(\frac{N}{I^{n-k}N}\right) + \lambda\left(\frac{L}{I^nL}\right)\right) =$$

$$= \max\left\{\deg\left(\lambda\left(\frac{N}{I^{n-k}N}\right)\right), \deg\left(\lambda\left(\frac{L}{I^nL}\right)\right)\right\} = \max\left\{\deg P_{I,N}, \deg P_{I,L}\right\}.$$

Therefore (1) follows. For (2) we have

$$\lambda\left(\frac{L}{I^nL}\right) + \lambda\left(\frac{N}{I^{n-k}N}\right) \leq \lambda\left(\frac{M}{I^nM}\right) \leq \lambda\left(\frac{L}{I^nL}\right) + \lambda\left(\frac{N}{I^nN}\right),$$

therefore the leading coefficients have to cancel, i.e.

$$\mathrm{LC}(\mathrm{P_{I,M}}) = \mathrm{LC}(\mathrm{P_{I,N}} + \mathrm{P_{I,L}}),$$

where LC denotes the leading coefficient of a polynomial. This is of course equivalent to (2). □

We are now ready to prove Theorem 7.

Proof of Theorem 7. Take a prime filtration of M and apply Lemma 8. Then, without loss of generality we can assume $M = R/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. Furthermore, passing from R to $R/annM = R/\mathfrak{p}$ we can directly assume that $M = R$ is a domain. Set $d = \dim R$ and choose a sop (system of parameters) $J = (x_1, \ldots, x_d) \subseteq I$. Then there exists $l \in \mathbb{N}$ such that $I^l \subseteq J$, and so

$$I^{ln} \subseteq J^n \subseteq I^n \quad \text{for all } n \in \mathbb{N}.$$

This implies

$$\lambda\left(\frac{R}{I^n}\right) \leq \lambda\left(\frac{R}{J^n}\right) \leq \lambda\left(\frac{R}{I^{ln}}\right)$$

and so

$$P_{I,R}(n) \leq P_{J,R}(n) \leq P_{I,R}(ln).$$

But $\deg P_{I,R}(n) = \deg P_{I,R}(ln)$ since $l \in \mathbb{N}$ is just a constant, therefore $\deg P_{J,R} = \deg P_{I,R}$. But we know that

$$\deg P_{I,R} = \deg P_{J,R} \leq \mu(J) = d.$$

Conversely, we induct on d. If $d = 0$ then clearly $\deg P_{I,R} \geq d$. Let $d > 0$ and pick $x \in \mathfrak{m}$, $x \neq 0$. Remember that R is a domain, hence we have a short exact sequence:

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow \overline{R} := R/xR \longrightarrow 0.$$

Notice that $\deg(P_{I,R} - P_{I,R} - P_{I,\overline{R}}) < \deg P_{I,R}$ by Lemma 8 (2), therefore

$$d - 1 = \dim \overline{R} = \deg P_{I,\overline{R}} < \deg P_{I,R},$$

that is $d \leq \deg P_{I,R}$. $\square$

Definition. Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local ring, let $\sqrt{I} = \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$. Then the multiplicity of M (with respect to I) is

$$e(I; M) := \lim_{n\to\infty} \frac{d!\ \lambda\left(M/I^n M\right)}{n^d}.$$

If $M = R$ we will denote $e(I) := e(I; R)$.

Notation. Given a function f we say that a function g is $O(f)$ if there exists a constant C such that $g(n) \leq Cf(n)$ for $n >> 0$.

Remark 9. $e(I; M) \in \mathbb{N}$ and $e(I; M) = 0$ if and only if $\dim M < \dim R$.

Proof. Set $s := \dim M$ and $d := \dim R$. In general $s \leq d$. Also

$$\lambda\left(\frac{M}{I^n M}\right) = \frac{b_s}{s!} n^s + O(n^{s-1})$$

with $b_s > 0$, and therefore

$$\frac{d!\ \lambda(M/I^n M)}{n^d} = \frac{d!\ b_s}{s!} n^{d-s} + O(n^{d-s-1}).$$

Taking the limit shows that

$$e(I; M) = \begin{cases} 0 & s < d \\ b_s & s = d \end{cases}$$

$\square$

Remark 10. Let $t \in \mathbb{N}$, $t \geq 1$. Then $e(I^t; M) = e(I; M)t^d$.

Proof. Assume $\dim M = \dim R = d$. We have seen that

$$\lambda\left(\frac{M}{I^n M}\right) = \frac{e(I; M)}{d!} n^d + O(n^{d-1}),$$

hence

$$\lambda\left(\frac{M}{(I^t)^n M}\right) = \frac{e(I; M)}{d!} (tn)^d + O(n^{d-1}) = \frac{e(I; M)t^d}{d!} n^d + O(n^{d-1}).$$

This means

$$e(I^t; M) = e(I; M)t^d.$$

If $\dim M < \dim R$, then the equality still holds since $e(I^t; M) = e(I; M) = 0$. $\square$

Remark 11. $e(I;M)$ is additive on short exact sequences because of Lemma 8 (2).

Remark 12. $e(I)=e(\hat{I})$, where $\hat{I}=I\hat{R}$, since $\sqrt{I}=\mathfrak{m}$ and so $R/I^n \simeq \hat{R}/\hat{I}^n$.

Theorem 9. *If $x_1,\dots,x_d$ is a regular sequence, then*

$$\mathrm{gr}_{(\underline{x})}R \simeq \frac{R}{(\underline{x})}[T_1,\dots,T_d].$$

Equivalently, $(\underline{x})^n/(\underline{x})^{n+1}$ is a free $R/(\underline{x})$-module of rank $\binom{n+d-1}{d-1}$.

For the proof see Corollary 22.

Remark 13. If $x_1,\dots,x_d$ is an sop for R as above and it is a regular sequence (this is equivalent to say that R is Cohen-Macaulay), then:

$$e((\underline{x}))=\lambda\left(\frac{R}{(\underline{x})}\right).$$

Moreover, if $x_1,\dots,x_d$ is just an sop, then

$$e((\underline{x}))\le\lambda\left(\frac{R}{(\underline{x})}\right).$$

Proof. By Theorem 9 we have

$$\lambda\left(\frac{R}{(\underline{x})^n}\right)=\sum_{i=0}^{n-1}\lambda\left(\frac{(\underline{x})^i}{(\underline{x})^{i+1}}\right)=\sum_{i=0}^{n-1}\lambda\left(\frac{R}{(\underline{x})}\right)\binom{i+d-1}{d-1}=$$
$$=\lambda\left(\frac{R}{(\underline{x})}\right)\binom{n+d}{d}=\lambda\left(\frac{R}{(\underline{x})}\right)\left[\frac{n^d}{d!}+O(n^{d-1})\right].$$

Hence

$$e((\underline{x}))=\lim_{n\to\infty}\frac{d!}{n^d}\lambda\left(\frac{R}{(\underline{x})}\right)=\lambda\left(\frac{R}{(\underline{x})}\right).$$

For the general inequality notice that in any case there exists a surjective ring homomorphism

$$\begin{gathered}B=R/(\underline{x})[T_1,\dots,T_d]\twoheadrightarrow \mathrm{gr}_{(\underline{x})}R=G\\ T_i\mapsto x_i+(\underline{x})^2\end{gathered}$$

and therefore

$$\lambda\left(\frac{R}{(\underline{x})}\right)=\lim_{n\to\infty}\frac{\lambda(B_n)(d-1)!}{n^{d-1}}\ge\lim_{n\to\infty}\frac{\lambda(G_n)(d-1)!}{n^{d-1}}.$$

In this case we multiplied by $(d-1)!$ because we know that if $\lambda(B/B_n)$ is eventually a polynomial of degree d, then $\lambda(B_n)$ is eventually a polynomial of degree $d-1$, and the leading coefficient, that gives the multiplicity, doesn't change. Notice now that $\dim R=\dim G$. In fact, more generally for $\sqrt{I}=\mathfrak{m}$,

$\lambda(I^n/I^{n+1})$ is eventually the Hilbert Polynomial of the associated graded ring $G = \mathrm{gr}_I R$ of degree $\dim G - 1$. But also, for $n >> 0$, $\sum_{i=0}^{n-1} \lambda(I^i/I^{i+1}) = \lambda(R/I^n) = P_{I,R}$ is a polynomial of degree both $\dim G$ by Remark 6 and $\dim R$ by Theorem 7. Therefore, back to our case:

$$\lambda\left(\frac{R}{(\underline{x})}\right) \geq \lim_{n\to\infty} \frac{\lambda(G_n)(d-1)!}{n^{d-1}} = e((\underline{x})).$$

The last inequality is again because $\lambda(R/(\underline{x})^n) = \sum_{i=0}^{n-1} \lambda((\underline{x})^i/(\underline{x})^{i+1})$ and so $\lambda((\underline{x})^n/(\underline{x})^{n+1}) = \lambda(G_n)$ is eventually a polynomial of degree $d-1$ that gives the multiplicity of $(\underline{x})$. □

Remark 14. Let $(R, \mathfrak{m}, k)$ be a RLR (regular local ring), then $e(\mathfrak{m}) = 1$.

Proof. By Remark 13 $e(\mathfrak{m}) = \lambda(R/\mathfrak{m}) = \dim_k k = 1$. □

Definition. $e(\mathfrak{m}) =: e(R)$ is often called the multiplicity of the ring R.

Remark 15. Let (R, m, k) be artinian, then $e(R) = \lambda(R)$.

Proof. It follow also by Remark 13, but there is also an easy direct proof. Since R is artinian we have $\mathfrak{m}^n = 0$ for $n >> 0$, hence

$$e(R) = \lim_{n\to\infty} 0!\ \lambda(R)n^0 = \lambda(R).$$

□

Example 9. Let $R = k[x,y]/(x^2, xy)$, then $\dim R = 1$. Also $\lambda(R_n) = 1$ for all $n \in \mathbb{N}$, therefore

$$e(R) = \lim_{n\to\infty} \frac{0!\ \lambda(R_n)}{n^0} = 1$$

but R is clearly not regular (it is not even Cohen-Macaulay). We will see that the converse to Remark 14, that is $e(R) = 1 \Rightarrow R$ is a RLR, holds if R is unmixed, and it is a theorem of Nagata.

Theorem 10 (Associativity formula)**.** *Let $(R, \mathfrak{m})$ be a local noetherian ring, let $\sqrt{I} = \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$. Then*

$$e(I; M) = \sum_{\substack{\mathfrak{p}\,\in\,\mathrm{Spec}R \\ \dim R/\mathfrak{p}\,=\,\dim R}} e(I; R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(M_\mathfrak{p}).$$

Proof. Take a prime filtration of M:

$$M_0 \subseteq M_1 \subseteq \ldots \subseteq M_s = M,$$

with $M_{i+1}/M_i \simeq R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Spec}R$. Multiplicity is additive on short exact sequences, hence

$$e(I; M) = \sum_{i=0}^{s-1} e(I; R/\mathfrak{p}_i).$$

We just have to count how many times each $\mathfrak{p}_i$ appears in the prime filtration. First notice that $e(I; R/\mathfrak{p}_i) = 0$ unless $\dim R/\mathfrak{p}_i = \dim R$, therefore

$$e(I; M) = \sum_{\{i:\ \dim R/\mathfrak{p}_i = \dim R\}} e(I; R/\mathfrak{p}_i).$$

Fix a $\mathfrak{p} \in \mathrm{Spec}R$ such that $\dim R/\mathfrak{p} = \dim R$. Localizing at $\mathfrak{p}$ we have

$$\left(\frac{R}{\mathfrak{p}_i}\right)_p = \begin{cases} 0 & \mathfrak{p}_i \neq \mathfrak{p} \\ (R/\mathfrak{p})_{\mathfrak{p}} = k(p) & \mathfrak{p}_i = \mathfrak{p} \end{cases}$$

since $\mathfrak{p}$ is minimal ($\dim R/\mathfrak{p} = \dim R$) and we cannot have $\mathfrak{p}_i \subsetneq \mathfrak{p}$. Here $k(p) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = (R/\mathfrak{p})_{\mathfrak{p}}$ is the residue field of the localzation $R_{\mathfrak{p}}$. Now, localizing at $\mathfrak{p}$ the filtration:

$$(M_0)_{\mathfrak{p}} \subseteq (M_1)_{\mathfrak{p}} \subseteq \ldots \subseteq M_{\mathfrak{p}}$$

gives us a composition series of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$, and its length is both $\lambda_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ and the times $\mathfrak{p}$ appears in the original filtration. Hence the associativity formula follows. $\square$

3 Superficial Elements

Let (R, m, k) be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a non zero element. We want to understand how do $e(R)$ and $e(R/(f))$ relate to each other. To do this we need several tools. the relation between $G = gr_m(R)$, $\bar{G} = gr_{\bar{m}}(\bar{R})$ where $\bar{R} = \frac{R}{f}$.

Definition. Let $I \subset R$ be an ideal. The Rees ring of I is defined to be

$$\mathcal{R}(I) := R \oplus I \oplus I^2 \oplus I^3 \ldots = \bigoplus_{n \geqslant 0} I^n.$$

Equivalently $\mathcal{R}(I) = R[It]$ as a subring of $R[t]$. Also, $\mathrm{Proj}(\mathcal{R}(I))$ is the blow up of $V(I)$ in $\mathrm{Spec}(R)$.

Remark 16. One can easily check that

$$gr_I(R) \simeq \mathcal{R}(I) \otimes_R R/I \simeq \frac{\mathcal{R}(I)}{I\mathcal{R}(I)}.$$

Definition. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let $I \subseteq R$ be an ideal. Let $f \in R$ be a non zero element. Since R is local and Noetherian we have $\bigcap_{n \geqslant 0} I^n = (0)$, therefore there exists $n \in \mathbb{N}$ such that $f \in I^n$, but $f \notin I^{n+1}$. Then the leading form of f in $\mathrm{gr}_I R$ is defined to be

$$f^* := [f] \in I^n/I^{n+1},$$

where $[f]$ denotes the equivalence class of f inside I^n/I^{n+1}.

Example 10. Consider $\mathfrak{m} = (x, y) \subseteq R = k[[x, y]]$, and $f = x^2 - y^3 \in R$. Then, $gr_m(R) \simeq k[[x, y]]$ and $f^* = (x^2 - y^3)^* = x^2 \in \mathfrak{m}^2/\mathfrak{m}^3$. Notice that

$$\frac{R}{(f)} = \frac{k[[x,y]]}{x^2 - y^3} \simeq k[[t^2, t^3]].$$

Set $\bar{R} := R/(f)$ and $\bar{\mathfrak{m}} := \mathfrak{m}/(f)$. One can prove that

$$gr_{\bar{\mathfrak{m}}}(\bar{R}) \simeq \frac{k[[x,y]]}{(x^2)} \simeq \frac{gr_{\mathfrak{m}}(R)}{(f^*)}.$$

Definition. Let $(R, \mathfrak{m}, K)$ be a local ring. Set $R(t) := R[t]_{\mathfrak{m}R[t]}$ and $\mathfrak{m}(t) := \mathfrak{m}R[t]_{\mathfrak{m}R[t]}$. Notice that

$$\frac{R(t)}{\mathfrak{m}(t)} \simeq k[t]_{k[t] \smallsetminus \{0\}} =: k(t),$$

therefore $(R(t), \mathfrak{m}(t), k(t))$ has infinite residue field. Also $R \to R(t)$ is a faithfully flat extension.

Definition. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and let $I \subseteq R$ be an ideal. Then an element $x \in I \smallsetminus I^2$ is said to be a superficial element for I (of degree 1) if there exists $c \in \mathbb{N}$ such that

$$(I^{n+1} : x) \cap I^c = I^n \quad \text{for all } n \geqslant c.$$

Proposition 11. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Assume that R contains k, and also that k is infinite. Let $I \subseteq R$ be an ideal. Then*

(i) Superficial elements exists

(ii) If x is superficial for I and furthermore x is a nonzero divisor in R, then

$$I^{n+1} : x = I^n \quad \textit{for all } n >> 0.$$

Proof. (i) Set $G := gr_I(R)$. Consider a partial primary decomposition of (0) in G:

$$(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \mathfrak{q}_l \cap J,$$

where $\mathfrak{q}_i$ are $\mathfrak{p}_i$-primary ideals such that $G_+ \not\subseteq \mathfrak{p}_i$ and J is the intersection of all components containing a power of G_+ (i.e. $\sqrt{J} = G_+$). Note that $\mathfrak{p}_i \cap G_1 \neq G_1$. Therefore we can choose $x \in I \setminus I^2$ such that $x^* \in \Gamma_{/1}$ = is not in $\mathfrak{p}_i$ for all $i = 1, \dots, l$.

Claim. x is superficial for I.

Proof of the Claim. Fix $c \in \mathbb{N}$ such that $(G_+)^c \subseteq J$, and notice that

$$\mathfrak{q}_1 \cap \dots \mathfrak{q}_l \cap (G_+)^c \subseteq \mathfrak{q}_1 \cap \dots \mathfrak{q}_l \cap J = (0).$$

Now we induct on $n \geqslant c$, and we want to prove that $(I^{n+1} : x) \cap I^c = I^n$ for $n \geqslant c$. Notice that we always have the inclusion $I^n \subseteq (I^{n+1} : x) \cap I^c$ for all $n \geqslant c$. For the other inclusion:

- If $n = c$ we clearly have the other inclusion as well.
- If $n > c$, let $y \in (I^{n+1} : x) \cup I^c$. Since $(I^{n+1} : x) \subseteq (I^n : x)$, we have

$$y \in (I^n : x) \cap I^c = I^{n-1},$$

where the last equality holds by induction. By way of contradiction assume $y \notin I^n$. Then looking at the initial forms of x and y we get

$$x^* \in I/I^2 = G_1 \quad \text{and} \quad y^* \in I^{n-1}/I^n = G_{n-1} \subset (G_1)^c.$$

Observe that $x^*y^* = 0$, since $xy \in I^{n+1}$. So

$$y^* \in (0 : x^*) = (\mathfrak{q}_1 : x^*) \cap \ldots \cap (\mathfrak{q}_l : x^*) \cap (J : x^*) \subseteq \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_l.$$

Therefore

$$y^* \in \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_l \cap (G_1)^c = (0),$$

which is a contradiction. So $y \in I^n$ and this proves the Claim.

(ii) Since x is a superficial element there exists $c \in \mathbb{N}$ such that

$$(I^{n+1} : x) \cap I^c = I^n.$$

In particular

$$I^n = (I^{n+1} : x) \cap I^c \subseteq (I^{n+1} : x).$$

Conversely, let $y \in (I^{n+1} : x)$. Then

$$xy \in I^{n+1} \cap (x) = ((x) \cap I^r)I^{n+1-r} \quad \text{for } n >> 0,$$

where the last equality follows from the Artin-Rees Lemma. Therefore $xy = xzi$ for some $i \in I^{n+1-r}$ and $xz \in (x) \cap I^r$. Since x is a nonzero divisor in R, we get

$$y = zi \in I^{n+1-r} \subset I^c \quad \text{for } n >> 0.$$

Hence $y \in (I^{n+1} : x) \cap I^c = I^n$. □

Corollary 12. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let $I \subseteq R$ be an ideal. let $x \in R$ be superficial for I. Set $G = gr_I(R)$ and $\bar{G} = gr_{I/(x)}(R/(x))$. Then there exits a natural map*

$$\frac{G}{(x^*)} \to \bar{G}$$

of degree 0, which is an isomorphism in all large degrees.

Proof. Since x is superficial there exists $c \in \mathbb{N}$ such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. Hence for all $n \geqslant 1$ we have

$$xI^{n-1} \subseteq x(I^n : x) = (x) \cap I^n.$$

Notice that clearly $(G/(x^*))_0 = R/I \simeq (\bar{G})_0$. For all $n \geqslant 1$ we have a surjection

$$\begin{aligned} \left(\frac{G}{(x^*)}\right)_n = \quad & \frac{I^n/I^{n+1}}{(xI^{n-1})/I^{n+1}} \simeq \\ & \simeq \frac{I^n}{xI^{n-1} + I^{n+1}} \twoheadrightarrow \frac{I^n}{((x) \cap I^n) + I^{n+1}} \simeq \\ & \simeq (\frac{I^n + (x)}{I^{n+1} + (x)}) \quad = \quad (\bar{G})_n. \end{aligned}$$

To get an isomorphism we need $xI^{n-1} \supseteq (x) \cap I^n$, since we have seen that the other inclusion always holds. By Artin-Rees Lemma, there exists $r \in \mathbb{N}$ such that for $n > r + c$

$$(x) \cap I^n = ((x) \cap I^r)I^{n-r} \subseteq xI^c.$$

Let $y \in (x) \cap I^n$, then $y = xa$ for some $a \in I^c$. But also $(x) \cap I^n = x(I^n : x)$, hence $y = xb$ for some $b \in (I^n : x)$. Therefore $a - b \in (0 : x) \subseteq (I^n : x)$, and hence

$$a \in I^c \cap (I^n : x) = I^{n-1}$$

by superficiality (since $n > r + c \geqslant c$). So, for $n >> 0$ (more precisely for $n > r + c$), the above map is an isomorphism. □

Example 11. Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $\dim R = n$. Let $G := gr_{\mathfrak{m}}(R) \simeq k[x_1, ..., x_n]$ and let $f \in \mathfrak{m}$ be an non zero element. Define

$$\operatorname{ord}(f) := \max\{n \in \mathbb{N} : f \in \mathfrak{m}^n\}.$$

Notice that $f^* \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Then

$$e(R/(f)) = \operatorname{ord}(f).$$

Proof. Notice that f^* is a nonzero divisor in G because it is a domain. Set $d := \operatorname{odd}(f)$. It is easy to prove that, being f^* a nonzero divisor, we have $\mathfrak{m}^N : f = \mathfrak{m}^{N-d}$ for all $N \geqslant d$. For $N \geqslant d$ consider the exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}^{N-d}}{\mathfrak{m}^N} \longrightarrow \frac{R}{\mathfrak{m}^N} \xrightarrow{f} \frac{R}{\mathfrak{m}^N} \longrightarrow \frac{R}{\mathfrak{m}^N + (f)} \longrightarrow 0,$$

so that $\lambda\left(\dfrac{R}{\mathfrak{m}^N + (f)}\right) = \lambda\left(\dfrac{\mathfrak{m}^{N-d}}{\mathfrak{m}^N}\right)$. But $\lambda\left(\dfrac{R}{m^N}\right) = \dbinom{N+n-1}{n}$, where

$n = \dim R$. Hence

$$\lambda\left(\frac{R}{m^N + (f)}\right) = \binom{N+n-1}{n} - \binom{N-d+n-1}{n} =$$

$$= \frac{(N+n-1)\cdot \ldots \cdot N}{n!} - \frac{(N-d+n-1)\cdot \ldots \cdot (N-d)}{n!} =$$

$$= \left\{\frac{N^n}{n!} + \frac{\binom{n}{2}N^{n-1}}{n!} - \left[\frac{N^n}{n!} + \frac{\left[\binom{n}{2} - dn\right]N^{n-1}}{n!}\right]\right\} + O(N^{n-2}) =$$

$$\frac{dnN^{n-1}}{n!} + O(N^{n-2}) = \frac{dN^{n-1}}{(n-1)!} + O(N^{n-2}).$$

Therefore the multiplicity is

$$e\left(\frac{R}{(f)}\right) = \lim_{N\to\infty} \frac{(n-1)!}{N^{n-1}}\lambda\left(\frac{R}{m^N + (f)}\right) = d.$$

□

Definition. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension $\dim R = d$. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then we can write

$$P_I(n) = \lambda\left(\frac{R}{\mathfrak{m}^n}\right) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d} + \ldots + (-1)^d e_d(I).$$

The integers $e_j(I)$, for $j = 0, \ldots, d$ are called the Hilbert coefficients of I. In particular $e_0(I) = e(I)$ is the multiplicity.

Proposition 13. *Let (R, m, k) be a Noetherian local ring, and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Let $x \in R$ be a superficial for I, which is also a nonzero divisor on R. Set $\bar{R} = R/(x)$ and $\bar{I} = I/(x)$. Then , ${e_j}^R(I) = {e_j}^{\bar{R}}(\bar{I})$ for all $0 \leqslant j \leqslant d-1$.*

Proof. Set $G := gr_I(R)$, and $\bar{G} := gr_{\bar{I}}(\bar{R})$. By Corollary 12, for $n >> 0$, there is a short exact sequence:

$$0 \longrightarrow G_n(-1) \xrightarrow{x^*} G_n \longrightarrow \bar{G}_n \longrightarrow 0.$$

Notice that

$$\sum_{j=0}^{n-1} \lambda(G_j) = \sum_{j=0}^{n-1} \lambda\left(\frac{I^j}{I^{j+1}}\right) = \lambda\left(\frac{R}{I^n}\right).$$

Therefore, using the short exact sequence above, for $n >> 0$ we get:

$$\lambda\left(\frac{R}{I^n}\right) - \lambda\left(\frac{R}{I^{n-1}}\right) = \sum_{j=0}^{n-1}\lambda(G_j) - \sum_{j=0}^{n-2}\lambda(G_j) = \sum_{j=0}^{n-1}\lambda(\bar{G}_j) + C = \lambda\left(\frac{\bar{R}}{\bar{I}^n}\right) + C,$$

where C is a constant that depends on the fact that the above sequence is exact only for $n >> 0$. Hence we have

$$\lambda\left(\frac{R}{I^n}\right) - \lambda\left(\frac{R}{I^{n-1}}\right) = \sum_{j=0}^{d} e_j(I)\binom{n-j+d}{d} - \sum_{j=0}^{d} e_j(I)\binom{n-1-j+d}{d} =$$

$$= \sum_{j=0}^{d-1} e_j(\bar{I})\binom{n-1-j+d}{d-1} + C.$$

Notice that

$$\binom{n-j+d}{d} - \binom{n-1-j+d}{d} = \binom{n-1-j+d}{d-1},$$

therefore we get the following equality of polynomials

$$\sum_{j=0}^{d} e_j(I)\binom{n-1-j+d}{d-1} = \sum_{j=0}^{d-1} e_j(\bar{I})\binom{n-1-j+d}{d-1} + C$$

which implies that all the coefficients have to be the same. In particular $e_j(I) = e_j(\bar{I})$ for all $j = 0, \ldots, d-1$. $\square$

Proposition 14. *Let $R \subseteq S$ be an extension of local Noetherian rings (or graded rings with $R_0 = k = S_0$) and $\mathfrak{m}_R$ (resp. $\mathfrak{m}_S$) be maximal ideals. (For the graded case, let $\mathfrak{m}_R = R_+$, and $\mathfrak{m}_S = S_+$.) Further, let I be an $\mathfrak{m}_R$-primary ideal in R (homogeneous in the graded case). Assume that R is a domain and that S is module finite over R. Let $k = R/\mathfrak{m}_R$, $L = S/\mathfrak{m}_S$, and F be the quotient field of R. Set*

$$r = \operatorname{rank}_R(S) = \dim_k S \otimes_R F.$$

Then,

$$e(IS;S) = \frac{e(I;R)\cdot r}{[L:k]}.$$

Lemma 15. *Given the notation in Proposition 14, if M is an S-module of finite length, then $\lambda_R(M) = \lambda_S(M)\cdot[L:k]$.*

Proof. Since length is additive and $\lambda_R(L) = [L:k]$, we have that

$$\lambda_R(M) = \lambda_S(M)\cdot\lambda_R(L) = \lambda_S(M)\cdot[L:k].$$

$\square$

Proof of Proposition 14. We compute

$$\lambda_S(S/I^nS) = \frac{e_S(IS;S)n^d}{d!} + O(n^{d-1})$$

where $d = \dim(S) = \dim(R)$. On the other hand,

$$\lambda_R(S/I^nS) = \lambda_S(S/I^nS) \cdot [L : k].$$

and

$$\lambda_R(S/I^nS) = \frac{e_R(I;S)n^d}{d!} + O(n^{d-1}).$$

Comparing terms gives us that

$$e_R(I;S) = e_S(IS;S)[L : k].$$

It remains to prove that $e_R(I;S) = r \cdot e_R(I;R)$.

To prove this, chose an F-basis of $S \otimes_R F$, say,

$$\frac{s_1}{1}, \ldots, \frac{s_r}{1}.$$

Consider the R submodule of S, $T = Rs_1 + \ldots + Rs_r$. Let ϕ be the surjection of R^r onto T, sending the i^{th} basis element to s_i for $i = 1, \ldots, r$. Because $S \otimes_R F = F^r$, $\phi \otimes 1$ is an isomorphism. Since $\ker(\phi)$ and $\operatorname{coker}(\phi)$ have smaller dimension than d, we know that

$$\begin{aligned} e_R(I;S) &= e_R(I;R^r) \\ &= r \cdot e_R(I;R). \end{aligned}$$

□

Theorem 16 (Nagata)**.** *Let (R, m, k) be a Noetherian, local, formally unmixed ring of dimension $\dim R = d$. Then $e(R) = 1$ if and only if R is regular.*

Proof. If R is regular, then clearly $e(R) = 1$. To prove the converse let us assume that R contains a field (even if the theorem holds in the general case). Also, we can assume that the residue field is infinite passing to $R(t) = R[t]_{\mathfrak{m}R[t]}$, and complete R, since the multiplicity doesn't change, and if $\widehat{R(t)}$ is regular, then R is regular as well. So without loss of generality R is a complete unmixed local ring with infinite residue field. Using Associativity formula we have

$$1 = e(R) = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \dim R/\mathfrak{p} = \dim R}} e(R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}).$$

Hence there exists a unique prime $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$, and since R is unmixed this is the only associated prime of R. Also $\lambda(R_\mathfrak{p}) = 1$ implies that $R_\mathfrak{p}$ is a field, and hence R is a domain. Since $|k| = \infty$, choose a minimal reduction $(x_1, \ldots, x_d)$ of $\mathfrak{m}$. Because R is complete, using Cohen structure theorem, we have a finite extension

$$S = k[[x_1, \ldots, x_d]] \subseteq R$$

and since both are domains, by Proposition 14, we have

$$1 = e(R) = e(S) \cdot \operatorname{rank}_S(R) = \operatorname{rank}_S(R),$$

since S is regular, and so $e(S) = 1$. Therefore $\operatorname{rank}_S(R) = 1$, and hence $R = S$ is regular. □

4 Integral Closure of Ideals

Definition. Let R be a ring and I an ideal of R. An element x is said to be integral over I if x satisfies a monic equation

$$x^n + i_1 x^{n-1} + \cdots + i_n = 0$$

such that $i_j \in I^j$. The set of all integral elements is called the integral closure of I and is denoted $\overline{I}$

Proposition 17. *The integral closure of an ideal is an ideal*

Proof. This is a corollary to Exercise 16 □

Example 12. Let $R = k[x, y]$ be the polynomial ring in two variables over a field k and $I = (x^2, y^2)$. Here we have that $xy \in \overline{I}$. To see this, notice that xy satisfies the polynomial $T^2 - x^2y^2 \in R[T]$. It is worth noting that $x^2y^2 \in I^2$.

Example 13. Let $R = \mathbb{C}[\![x_1, \ldots, x_n]\!]$ be the power series in n variables over $\mathbb{C}$ and let $f \in R$ such that $f(0) = 0$. Then f is always integral over its partial derivatives

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Example 14 (Dedekind-Mertens)**.** Let R be a commutative ring and f, g two elements of $R[t]$, i.e.

$$f(t) = a_n t^n + \cdots + a_0;$$
$$g(t) = b_m t^m + \cdots + b_0,$$

If I is the content of the product fg, then $a_i b_j$ is integral over I for all i, j. An example of this is the following: let both f and g have degree one. We have that I is the ideal generated by the coefficients of the product $(a_1 t + a_0)(b_1 t + b_0)$. That is,

$$I = (a_1 b_1, a_0 b_0, a_1 b_0 + a_0 b_1).$$

Notice that

$$(a_1 b_0)^2 = (a_1 b_0 + a_0 b_1)(a_1 b_0) - (a_1 b_1)(a_0 b_0).$$

As $(a_1 b_0 + a_0 b_1) \in I$ and $(a_1 b_1)(a_0 b_0) \in I^2$, we have that $a_1 b_0$ satisfies a degree two monic polynomial in $R[T]$.

Open Question. *In the context Example 13, is* $f \in \mathfrak{m}\overline{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)}$*?*

To see an example of this question, let $f = x^3 + y^4$ in the ring $R = \mathbb{C}[\![x, y]\!]$. Then

$$\frac{\partial f}{\partial x} = 3x^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y^3.$$

We thus see that $x^3 + y^4 \in \mathfrak{m}(x^2, y^3) \subseteq \mathfrak{m}\overline{(x^2, y^3)}$.

Proposition 18. *Let R be a noetherian ring and $J \subseteq I$ ideals in R. The following are equivalent:*

(1) $I \subseteq \overline{J}$;

(2) $\mathcal{R}(I)$ *is module-finite over* $\mathcal{R}(J)$;

(3) $I^n = JI^{n-1}$ *over all* $n >> 0$;

(4) there exists a k *such that* $I^n \subseteq J^{n-k}$ *for all* $n \geqslant k$.

Further, if R is also local, the above are also equivalent to

(5) Let

$$\mathcal{F}_I := \frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} = R/m \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots$$

and define A to be the subring of $\mathcal{F}_I$ generated by

$$\frac{J + \mathfrak{m}I}{\mathfrak{m}I} \subseteq I/\mathfrak{m}I$$

over $R/\mathfrak{m}$. Then $\mathcal{F}_I$ is module-finite over A.

Proof. (1) $\Leftrightarrow$ (2): Use Exercise 16 and the fact that $\mathcal{R}(I)$ is a finitely generated as a ring over $\mathcal{R}(J)$ since I is finitely generated.

(3) $\Rightarrow$ (4): Fix k such that $I^n = JI^{n-1}$ for all $n \geqslant k$. By induction,

$$I^{k+l} = JI^{k+l-1} = J^2 I^{k+l-2} = \cdots = J^{l+1} I^{k-1} \subset J^{l+1}.$$

(4) $\Rightarrow$ (2): Part (4) gives us that

$$\mathcal{R}(J) \subseteq \mathcal{R}(I) \subseteq \mathcal{R}(J) \cdot \frac{1}{t^k}.$$

since $\mathcal{R}(j)$ is noetherian, we have that $\mathcal{R}(I)$ is finitely generated as a $\mathcal{R}(J)$-module.

(2) $\Rightarrow$ (3): As $\mathcal{R}(I)$ is module-finite over $\mathcal{R}(J)$, consider

$$\mathcal{R}(I) = R \oplus It \oplus I^2 t^2 \oplus \cdots \oplus I^N t^N \oplus \cdots$$
$$\mathcal{R}(J) = R \oplus Jt \oplus J^2 t^2 \oplus \cdots \oplus J^N t^N \oplus \cdots .$$

Say the homogeneous generators $u_1, \ldots, u_s$ are up to degree N in t, that is, $\deg(u_i) = d_i$ where $d_i \leqslant N$ and

$$\mathcal{R}(I) = \mathcal{R}(J) u_1 + \cdots + \mathcal{R}(J) u_s.$$

So, we have that

$$I^L t^L \subseteq (J^{L-d_1} t^{L-d_1}) u_1 + \cdots + (J^{L-d_s} t^{L-d_s}) u_s.$$

Since $u_1 1 \in I_i^d t^i$ for all $i = 1, \dots, s$, we see that

$$I^L \subseteq J^{L-d_1} I^{d_1} + \cdots + J^{L-d_s} I^{d_s}.$$

But $J \subseteq I$, so for L larger than $\max\{d_k\}$,

$$I^L \subseteq JI^{L-1} \subseteq I^L.$$

(2) $\Rightarrow$ (18): Assume that $(R, \mathfrak{m})$ is local. As $\mathcal{R}(J) \subseteq \mathcal{R}(I)$ is module-finite, we have that

$$A = \frac{\mathcal{R}(J)}{\mathfrak{m}\mathcal{R}(I) \cap \mathcal{R}(J)} \subseteq \frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} = \mathcal{F}_I$$

is module-finite as well.

(18) $\Rightarrow$ (3): Assume that $(R, \mathfrak{m})$ is local. Say $\mathcal{F}_I$ is generated over A by a finite number of elements up to degree k. Just as in the proof of (2) $\Rightarrow$ (3), this means for $n > k$,

$$\frac{I^n}{\mathfrak{m}I^n} = \frac{JI^{n-1} + \mathfrak{m}I^n}{\mathfrak{m}I^n}.$$

Thus, we have that $I^n = JI^{n-1} + \mathfrak{m}I^n$ and hence by Nakayama's lemma we see that $I^n = JI^{n-1}$. □

Definition. A local noetherian ring $(R, \mathfrak{m}, k)$ is formally equidimensional if for all minimal primes $\mathfrak{p}$ of $\widehat{R}$, $\dim(\widehat{R}/\mathfrak{p}) = \dim(\widehat{R})$.

Theorem 19 (Rees)**.** *Let $(R, \mathfrak{m}, k)$ be a formally equidimensional local noetherian ring and $J \subset I$ $\mathfrak{m}$-primary ideals. Then $e(J) = e(I)$ if and only if $I \subset \overline{J}$.*

Remark 17. The fact that $I \subseteq \overline{J}$ implies equality of multiplicities does not require formally equidimesional.

Remark 18. We will prove Theorem 19 only when R contains an infinite field. The theorem is true otherwise, but is omitted from these notes.

Before we can prove Theorem 19, we will need the following lemma.

Lemma 20. *Let $(R, \mathfrak{m})$ be a local noetherian ring of dimension d and suppose that $J \subseteq I$. Then, $I \subseteq \overline{J}$ if and only if for all minimal primes $\mathfrak{p}$ in R,*

$$\frac{I + \mathfrak{p}}{\mathfrak{p}} \subseteq \overline{\frac{J + \mathfrak{p}}{\mathfrak{p}}}.$$

Proof. If we assume that $I \subseteq \overline{J}$, we can use the same integral equation to obtain the desired result.

Conversely, fix i in I and consider the multiplicatively closed subset of R,

$$W = \{f(i) \mid f(t) = t^m + j_1 t^{m-1} + \cdots + j_m\}.$$

Let $\operatorname{Rad}(R)$ denote the nilradical of R, that is, the intersection of all prime ideals in R. If $W \cap \operatorname{Rad}(R) \neq \emptyset$, then let $f(i)$ be an element of $\operatorname{Rad}(R)$. Thus there exists an N such that $f(i)^N = 0$ and hence i is integral over J.

If $W \cap \operatorname{Rad}(R) = \emptyset$, then there exists a prime $\mathfrak{q}$ such that $\mathfrak{q} \cap W = \emptyset$. Therefore there exists a minimal prime $\mathfrak{p}$ in R such that $\mathfrak{p} \cap W = \emptyset$. Thus, i is not an element of $\overline{J + \mathfrak{p}}/\mathfrak{p}$, a contradiction. □

Proof of Theorem 19. First assume that $I \subseteq \overline{J}$. Thus there exists an l such that for all $n \geqslant l$,

$$J^n \subseteq I^n \subseteq J^{n-l} \subseteq I^{n-l}.$$

Therefore

$$\lambda(R/J^{n-k}) \subseteq \lambda(R/I^n) \subseteq \lambda(R/J^n)$$

and hence we have that $e(I) = e(J)$.

Now assume that $e(I) = e(J)$ and that R contains an infinite filed. Not that $e(I\widehat{R}) = e(J\widehat{R})$. Further, by Proposition 18, $I\widehat{R} \subset \overline{J\widehat{R}}$ if and only if there exists an l such that $I^n\widehat{R} \subseteq J^{n-k}\widehat{R}$ for all $n \geqslant l$. Thus,

$$I^n = I^n\widehat{R} \cap R \subseteq J^{n-l}\widehat{R} \cap R = J^{n-l}.$$

Applying the proposition once again yields the fact that $I \subseteq \overline{J}$. So, without losing any generality, $R = \widehat{R}$.

Using the associativity formula (Theorem 10),

$$e(I) = \sum e(I; R/\mathfrak{p})\lambda(R_\mathfrak{p});$$
$$e(J) = \sum e(J; R/\mathfrak{p})\lambda(R_\mathfrak{p}),$$

where $\mathfrak{p}$ is a minimal prime in R such that $\dim(R/\mathfrak{p}) = \dim(R)$. The fact that $J \subseteq I$ shows us $e(J; R/\mathfrak{p}) \geqslant e(I; R/\mathfrak{p})$. However, since R is formally equidimensional, we have equality for all minimal primes $\mathfrak{p} \in R$. By Lemma 20, if

$$\frac{I+\mathfrak{p}}{\mathfrak{p}} \subseteq \overline{\frac{J+\mathfrak{p}}{\mathfrak{p}}}$$

for all minimal primes $\mathfrak{p}$ in R, then we have that $I \subseteq \overline{J}$. Hence, we may assume R is a complete local domain.

We now turn to a reduction of the ideals I and J. We can replace J by a system of parameters $(x_1, \ldots, x_d) \subseteq J$. It is enough to show there exists a system of parameters $(x_1, \ldots, x_d) \subseteq J$ such that $J \subseteq \overline{(x_1, \ldots, x_d)}$. If so, then we know by the easy direction that

$$e(x_1, \ldots, x_d) = e(I).$$

But then, $e(x_1, \ldots, x_d) = e(I)$. If we prove that $I \subseteq \overline{(x_1, \ldots, x_d)}$ then we are able to deduce that $I \subseteq \overline{J}$.

To do this, use Noether normalization on the fiber ring

$$\mathcal{F}_J = \frac{R}{\mathfrak{m}} \oplus \frac{J}{\mathfrak{m}J} \oplus \frac{J^2}{\mathfrak{m}J^2} \cdots .$$

Keep in mind that $\mathcal{F}_J$ is a finitely generated k algebra. By Noether's normalization,

$$k[x_1^*, \ldots, x_l^*] \subseteq \mathcal{F}_J,$$

where x_i^* are elements of $J/\mathfrak{m}J$. But lifting x_i^* to $x_i \in J$, we may apply Proposition 18 to see that $J \subseteq \overline{(x_1, \dots, x_l)}$.

Note that $\overline{(x_1, \dots, x_l)} \subseteq \sqrt{(x_1, \dots, x_l)}$ and hence $(x_1, \dots, x_l)$ is $\mathfrak{m}$-primary (this follows since $\sqrt{J} = \mathfrak{m}$). Therefore, $lgsd$ by Krull's height theorem. But,

$$l = \dim(\mathcal{F}_J) \leqslant \dim(\mathrm{gr}_J R) = d,$$

and hence $l = d$. (To see this last fact, notice that $\mathrm{gr}_J R/\mathfrak{m}\mathrm{gr}_J R \simeq \mathcal{F}_J$.)

Thus, without losing any generality, we may assume that J is generated by the system of parameters $(x_1, \dots, x_d)$. Next we make I as simple as possible. Choose any $y \in I$. It is enough to show that $y \in \overline{J}$. Thus, we can replace I by $(x_1, \dots, x_d, y)$. Note that in this case,

$$e(x_1, \dots, x_d) = e(x_1, \dots, x_, y).$$

We are now able to make a further reduction on the rings. Since $k \subseteq R$, by Cohen's structure theorem, we can consider the extenstion

$$\begin{array}{ccc} R & \supseteq & B = k[\![x_1, \dots, x_d, y]\!] \\ | & & | \\ k[\![x_1, \dots, x_d]\!] & = & A \end{array}$$

where R is finite over $k[\![x_1, \dots, x_d]\!]$. We may assume that $R = B$. To see this, let the maximal ideal of B be $\eta = (x_1, \dots, x_d, y)B$ and let $r = \mathrm{rank}_B(R)$. We know that

$$\begin{aligned} e_R(I) &= e_R(\eta R) = e_B(\eta) \cdot r \\ e_R(J) &= e_R((x_1, \dots, x_d)R) = e_B(x_1, \dots, x_d) \cdot r \end{aligned}$$

and hence we have that $e_R(x_1, \dots, x_d) = e_B(\eta)$. So, if we prove the theorem is true for $R = B$, we get that $y \in \overline{(x_1, \dots, x_d)B}$ which implies that $y \in \overline{(x_1, \dots, x_d)R}$.

Now, $R = k[\![x_1, \dots, x_d, y]\!]$ is a complete domain of dimension d, $x_1, \dots, x_d$ is an system of parameters, $J = (x_1, \dots, x_d)$, and I is the maximal ideal $\mathfrak{m}$. Note that

$$R \simeq k[\![x_1, \dots, x_d, T]\!]/\mathfrak{p}$$

where $\mathfrak{p}$ is a height one prime. But $k[\![x_1, \dots, x_d, T]\!]$ is a unique factorization domain and hence

$$\mathfrak{p} = (T^l + a_1 T^{l-1} + \cdots + a_l) =: (f).$$

Now, $e(\mathfrak{m}) = \mathrm{ord}(f)$ and $e(x_1, \dots, x_d) = \lambda(R/(x_1, \dots, x_d))$. As

$$\frac{R}{(x_1, \dots, x_d)} = \frac{k[\![T]\!]}{T^l},$$

we have that $e(x_1, \dots, x_d) = l$ (here we are assuming that $l = \mathrm{ord}(f)$). This implies that $a_i \in (x_1, \dots, x_d)^i$, otherwise $\mathrm{ord}(f) < l$. Since $f(y) = 0$, this shows that $y \in \overline{(x_1, \dots, x_d)}$. $\square$

5 Associated Graded Ring and Rees Algebra

Throughout this section let R be a noetherian ring. Let $I \subseteq R$ be an ideal, then we have already defined

$$G := \mathrm{gr}_I R = \bigoplus_{n \geq 0} I^n / I^{n+1}$$

the associated graded ring of R with respect to I. We have also defined

$$\mathcal{R}(I) \simeq R[It] = \bigoplus_{n \geq 0} I^n t^n \subseteq R[t]$$

the Rees ring of I. Notice that

$$G \simeq \frac{\mathcal{R}(I)}{I\mathcal{R}(I)}$$

and often is more convenient to study $\mathcal{R}(I)$ instead of $\mathrm{gr}_I R$.

5.1 Equations defining Rees Algebras

Let $I = (x_1, \dots, x_n) \subseteq R$. Then there exists a graded surjective map

$$\begin{aligned} \varphi : R[T_1, \dots, T_n] &\twoheadrightarrow \mathcal{R}(I) \\ T_i &\mapsto x_i t \end{aligned}$$

Hence $\ker \varphi \subseteq R[T_1, \dots, T_n]$ is a homogeneous ideal, and if R itself is graded, then $\ker \varphi$ is bigraded.

Remark 19. Set $\mathfrak{a} := \ker \varphi$. Then $\mathfrak{a}$ is homogeneous and it is generated by homogeneous polynomials $F(T_1, \dots, T_n)$ (say $\deg F = d$) such that

$$0 = \varphi(F(T_1, \dots, T_n)) = F(x_1 t, \dots, x_n t) = t^d F(x_1, \dots, x_n),$$

that is $\mathfrak{a} = (F \in R[T_1, \dots, T_n] : F(x_1, \dots, x_n) = 0)$.

Example 15. Let $R = k[x, y]$ and let $I = (x, y)^2 = (x^2, xy, y^2)$. Then

$$\mathfrak{a} = (yT_1 - xT_2, yT_2 - xT_3, T_2^2 - T_1 T_3).$$

Definition. Let $\mathfrak{a}_i$ = "the ideal generated by all homogeneous polynomial of degree at most i".

In the previous example $\mathfrak{a}_1 = (yT_1 - xT_2, yT_2 - xT_3)$ and $\mathfrak{a}_2 = \mathfrak{a}$.

Remark 20. Since R is noetherian, there exists $N >> 0$ such that $\mathfrak{a}_N = \mathfrak{a}$, since $\mathfrak{a}$ is finitely generated.

Definition. I is said to be linear type if $\mathfrak{a} = \mathfrak{a}_1$.

Example 16. Present $I = (x_1, \ldots, x_n) \subseteq R$:

$$R^m \xrightarrow{A} R^n \longrightarrow I \longrightarrow 0$$

$$e_i \longrightarrow x_i$$

and let $A = (a_{ij})$, so that $\sum_i x_i a_{ij} = 0$. Then it is easy to prove that

$$\mathfrak{a}_1 = \left(\sum_{i=1}^{n} T_i a_{ij} : j = 1, \ldots, n \right).$$

Definition. A sequence of elements $x_1, \ldots, x_n$ is said to be a d-sequence if for all $i \geq 0$

$$((x_1, \ldots, x_i) : x_{i+1}) \cap (x_1, \ldots, x_n) = (x_1, \ldots, x_i),$$

where we set $x_0 = 0$.

Example 17. Any regular sequence is a d-sequence since for all $i \geq 0$

$$(x_1, \ldots, x_i) : x_{i+1} = (x_1, \ldots, x_i).$$

Remark 21. If $n = 1$, then $x_1 = x$ is a d-sequence if and only if

$$0 : x = 0 : x^2.$$

In fact let x be a d-sequence, then $(0 : x) \cap (x) = 0$. Clearly $0 : x \subseteq 0 : x^2$. Let $ax^2 = 0$, then $ax \in (0 : x) \cap (x) = 0$, that is $a \in 0 : x$. Conversely let $0 : x = 0 : x^2$, and let $bx \in (0 : x) \cap (x)$. Hence $bx^2 = 0$, that is $b \in 0 : x^2 = 0 : x$. So $bx = 0$ and x is a d-sequence.

Example 18. Let $R = k[x, y]/(x^2, xy)$. Then $y \in R$ is not a regular sequence, but it is a d-sequence since $0 : y = 0 : y^2 = xR$.

Definition. Let $F \in R[T_1, \ldots, T_n]$ be a homogeneous polynomial. Then we say that F has weight j if

$$F \in (T_1, \ldots, T_j) \setminus (T_1, \ldots, T_{j-1}).$$

We denote $j = wt(F)$.

Example 19. Let $F = T_1^2 + T_1 T_2 + T^4 + T_4 T_3 + T_2 T_6 \in R[T_1, \ldots, T_6]$. Then $wt(F) = 4$ since

$$F \in (T_1, \ldots, T_4) \setminus (T_1, \ldots, T_3).$$

Theorem 21. *Let $x_1, \ldots, x_n$ be a d-sequence and let $I = (x_1, \ldots, x_n)$. Then I is linear type.*

Proof. By induction on both degree and weight we will prove the following

Claim. If $F(T_1, \ldots, T_n)$ is homogeneous of degree $d \geq 1$ is such that $F(x_1, \ldots, x_n) \in (x_1, \ldots, x_j)R$, then there exists $G(T_1, \ldots, T_n)$ a form of degree d and weight at most j such that $F - G \in \mathfrak{a}_1$.

Proof of the Claim. Assume $d = 1$. By assumption there exists $r_i \in R$, for $1 \leq i \leq j$, such that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{j} r_i x_i.$$

But $\deg F = 1$, so set

$$G := \sum_{i=1}^{j} r_i T_i,$$

so that $(F - G)(x_1, \ldots, x_n) = 0$, that is $F - G \in \mathfrak{a}_1$ and clearly $wt(G) \leq j$.

Assume now $d > 1$ and induct on the weight of F. If $wt(F) \leq j$, then set $G = F$, so that clearly $F - G \in \mathfrak{a}_1$. Then suppose $wt(F) = k > j$ and write

$$F = T_k F_1 + F_2,$$

where $\deg F_2 = \deg F_1 + 1 = d$ and $wt(F_2) < k$. Then

$$F(x_1, \ldots, x_n) = x_k F_1(x_1, \ldots, x_n) + F_2(x_1, \ldots, x_n) \in (x_1, \ldots, x_j)R \subseteq (x_1, \ldots, x_{k-1})R.$$

Since $F_2(x_1, \ldots, x_n) \in (x_1, \ldots, x_{k-1})$ we have that

$$F_1(x_1, \ldots, x_n) \in ((x_1, \ldots, x_{k-1}) : x_k) \cap (x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1})$$

since $x_1, \ldots, x_n$ is a d-sequence. By induction on the degree there exists G_1 of degree $d - 1$ with $wt(G_1) \leq k - 1$ and such that $F_1 - G_1 \in \mathfrak{a}_1$. Set

$$F' := T_k G_1 + F_2.$$

Then $F - F' = T_k(F - G_1) \in \mathfrak{a}_1$. Note that $F(x_1, \ldots, x_n) = F'(x_1, \ldots, x_n) \in (x_1, \ldots, x_j)$ and $wt(F') \leq k - 1$ since $wt(G_1), wt(F_2) \leq k - 1$. By induction on weight there exists G homogeneous of degree d, with $wt(G) \leq j$, such that $F' - G \in \mathfrak{a}_1$. Hence

$$F - G = (F - F') + (F' - G) \in \mathfrak{a}_1.$$

Finally, the Claim implies the theorem with $j = 0$: let F be a non zero homogeneous polynomial of degree $d \geq 1$ such that $F \in \mathfrak{a}$, i.e. $F(x_1, \ldots, x_n) = 0$. Then there exists G homogeneous of degree d, with $wt(G) \leq 0$, such that $F = G \in \mathfrak{a}_1$. But $wt(G) = 0$, $\deg G = d \geq 1$ implies that $G = 0$. Therefore $F \in \mathfrak{a}_1$. $\square$

Corollary 22. *Let* $(R,\mathfrak{m})$ *be a noetherian local ring and let* $x_1,\ldots,x_n$ *be a regular sequence. Then*

$$\operatorname{gr}_{(\underline{x})}R \simeq \frac{R}{(\underline{x})}[T_1,\ldots,T_n].$$

Proof. If $x_1,\ldots,x_n$ is a regular sequence, then it is a d-sequence. Also $\mathfrak{a}_1$ is generated by the Koszul relations $x_iT_j - x_jT_i$ by Example 16. Therefore

$$\mathcal{R}((\underline{x})) = \frac{R[T_1,\ldots,T_n]}{2\times 2\begin{pmatrix} x_1\ldots x_n \\ T_1\ldots T_n\end{pmatrix}}$$

Hence

$$\operatorname{gr}_{(\underline{x})}R \simeq \frac{\mathcal{R}((\underline{x}))}{(\underline{x})\mathcal{R}((\underline{x}))} \simeq \frac{R[T_1,\ldots,T_n]}{(\underline{x}) + 2\times 2\begin{pmatrix} x_1\ldots x_n \\ T_1\ldots T_n\end{pmatrix}} \simeq \frac{R}{(\underline{x})}[T_1,\ldots,T_n].$$

□

6 Exercises

(1) Assume R is a noetherian and that M, N are finitely generated graded R-modules. Prove that Hom

(2) Suppose that $S = k[x_1, \ldots, x_n]$ with the usual grading. Let $f_1, \ldots, f_r$ be a regular sequence with $\deg(f_i) = d_i$. Find $H_{S/(f_1,\ldots,f_r)}(t)$.

(3) If $R = k[x_1, \ldots, x_n]$ is a polynomial ring in n variables with $\deg(x_i) = d_i$, then

$$H_R(t) = \frac{1}{\prod(1 - t^{d_i})}.$$

(4) Let $R = \oplus_{i\geqslant 0} R_i$ be a graded noetherian ring. Prove there exists and N such that $R_{NK} = (R_N)^K$ for all $K > 0$. That is, the subring

$$R_0 \oplus R_N \oplus R_{2N} \oplus \cdots$$

is generated in degree N.

(5) Let R be as in Exercise 4. Then R is integral and finite over $\oplus_{i\geqslant 0} R_{Ni}$.

(6) Let $R = k[x_1, \ldots, x_n]$ with $\deg(x_i) = k_i$. What is the least N that satisfies Exercise 4?

(7) Assume G is a noetherian graded ring, $(G_0, \mathfrak{m}_0)$ is artinian local and $G = G_0[G_1]$. Let $\mathfrak{m} := \mathfrak{m}_0 \oplus G_1 \oplus \ldots$ as G-module. Then clearly $G/\mathfrak{m} \simeq G_0/\mathfrak{m}_0$ is a field, and so $\mathfrak{m}$ is maximal in G. Let $R := G_{\mathfrak{m}}$. Prove that

$$\mathrm{gr}_{\mathfrak{m}} R = G.$$

(n.b. the fact that G is generated in degree one is crucial).

(8) Let $(R, \mathfrak{m})$ be a noetherian local ring and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then

$$\mathrm{gr}_I R \simeq \mathrm{gr}_{I\hat{R}} \hat{R}.$$

(9) Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Prove that:

- $\mathrm{gr}_I R$ is a domain $\Rightarrow$ R is a domain.
- $\mathrm{gr}_I R$ is integrally closed $\Rightarrow$ R is integrally closed.
- $\mathrm{gr}_I R$ is CM (Cohen-Macaulay) $\Rightarrow$ R is CM.
- $\mathrm{gr}_I R$ is Gorenstein $\Rightarrow$ R is Gorenstein.

Is it true that if $\mathrm{gr}_I R$ is a UFD then R is a UFD?

(10) Find an example (with prove) of a local domain such that the completion (with respect to the maximal ideal) is not a domain. (Hint: By Exercises 8 and 9, if $\mathrm{gr}_{\mathfrak{m}} R \simeq \mathrm{gr}_{\hat{\mathfrak{m}}} \hat{R}$ is a domain, then $\hat{R}$ is a domain, so a possible example has to be such that $\mathrm{gr}_{\mathfrak{m}} R$ is not a domain.)

(11) Which Artinian local rings A with residue field $\mathbb{C}$, can be embedded in $\frac{\mathbb{C}[|t|]}{(t^n)}$.

(12) If $\sqrt{I} = m$, then $e(I, R) = e(IR(t), R(t))$.

(13) Complete the proof of the fact that superficial element exits by proving there exists $x \in I \setminus I^2$ such that x^* is not in p_i for all $i = 1, 2, ..., l$.

(14) $I \subset R$, $f \in R$. Suppose $f \in I^d \setminus I^{d+1}$.TFAE:

   (a) For all $n \geq d$, $I^n : f = I^{n-d}$.
   (b) f^* is a NZD in $gr_I(R)$.

(15) (R, m, k) local $f \in m^d \setminus m^{d+1}$. Assume f^* is NZD in $gr_m(R)$. Then $e(R/(f)) = de(R)$.

(16) Let $J \subseteq I$ be ideal in a commutative ring R. Then $I \subseteq \overline{J}$ if and only if $\mathcal{R}(I)$ is integral over $\mathcal{R}(J)$.

Chapter 2

Grothendieck Groups

Throughout this chapter R will always be a noetherian ring. Define $H(R)$ to be the free abelian group on the isomorphism classes of finitely generated R-modules. Given $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ denote $< M >$ its class inside $H(R)$, that is the generator in $H(R)$ corresponding to the isomorphism class of M. Define also $L(R)$ to be the subgroup of $H(R)$ generated by elements

$$\{< M > - < M_1 > - < M_2 >\}$$

for which there exists a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$. Finally we define the Groethendieck Group of R to be

$$G_0(R) := \frac{H(R)}{L(R)}.$$

Give $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ write $[M]$ for its class inside $G_0(R)$.

Remark 22. If we have two short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

and

$$0 \longrightarrow M_1 \longrightarrow N \longrightarrow M_2 \longrightarrow 0,$$

then we have $[M] = [N]$ in $G_0(R)$.

Remark 23. If we restrict to projective module and we do the same costruction, we get the K-group $K_0(R)$.

Question. *Given $M, N \in \mathrm{Mod}^{\mathrm{fg}}(R)$ when is $[M] = [N]$?*

1 Basic Lemmas and Remarks

Lemma 23 (Filtration Lemma)**.** *Let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ and suppose*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

is a filtration. Then

$$[M] = \sum_{i=0}^{n-1} \left[\frac{M_{i+1}}{M_i} \right].$$

Proof. By induction on n. The step $n = 1$ is trivial. Let $n > 1$ and consider

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow \frac{M}{M_1} \longrightarrow 0,$$

so that $[M] = [M_1] + [M/M_1]$. Then we have a filtration

$$0 = \frac{M_1}{M_1} \subseteq \frac{M_2}{M_1} \subseteq \ldots \subseteq \frac{M_n}{M_1} = \frac{M}{M_1}$$

of length $n-1$. By induction we get

$$[M] = [M_1] + [M/M_1] = \sum_{i=0}^{n-1} \left[\frac{M_{i+1}}{M_i} \right].$$

□

Corollary 24. *$G_0(R)$ is generated by $[R/\mathfrak{p}]$, for $\mathfrak{p} \in \mathrm{Spec} R$.*

Proof. Take a prime filtration of M and apply the Filtration Lemma. □

Lemma 25 (Long Exact Sequence Lemma)**.** *Given an exact sequence*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_0 \longrightarrow 0,$$

then

$$\sum_{i=0}^{n} (-1)^n [M_i] = 0.$$

Proof. By induction again. □

Lemma 26 (Additive Map Lemma)**.** *Suppose $\varepsilon : \mathrm{Mod}^{\mathrm{fg}}(R) \to \mathbb{Z}$ is an additive function on short exact sequences. Then there exists an induced homomorphism $\widetilde{\varepsilon} : G_0(R) \to \mathbb{Z}$*

Proof. Given $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$, the homomorphism $\widetilde{\varepsilon}$ is defined to be

$$\widetilde{\varepsilon}(< M >) = \varepsilon(M)$$

on elements of the basis of $H(R)$. Since ε is additive on short exact sequences we have $\widetilde{\varepsilon}(L(R)) = 0$, hence there is an induced map $\widetilde{\varepsilon} : G_0(R) \to \mathbb{Z}$. □

Example 20. Let R be a PID. Then

$$G_0(R) \simeq \mathbb{Z}$$

generated by $[R]$.

Proof. By Corollary 24 we can consider just $[R/\mathfrak{p}]$, with $\mathfrak{p}$ prime in R. If $\mathfrak{p} = 0$, then $[R/\mathfrak{p}] = [R]$. If $\mathfrak{p} \neq 0$, then $\mathfrak{p} = (x)$ for some $x \neq 0$, hence the following sequence is exact

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow R/\mathfrak{p} \longrightarrow 0,$$

so that in $G_0(R)$ we have $[R/\mathfrak{p}] = 0$. Therefore $G_0(R) \simeq \mathbb{Z}[R]$, but we still have to prove that $[R]$ is not a torsion element. Consider the rank function. By Additive Map Lemma there exists a homomorphism

$$\begin{aligned} G_0(R) &\longrightarrow \mathbb{Z} \\ [M] &\longmapsto \widetilde{\mathrm{rank}}([M]) \end{aligned}$$

which is surjective since $[R] \mapsto 1$. So $G_0(R)$ cannot be torsion and hence $G_0(R) = \mathbb{Z}[R] \simeq \mathbb{Z}$. □

Example 21. Let $(R, \mathfrak{m}, k)$ be a regular local ring. Then $G_0(R) = \mathbb{Z}[R] \simeq \mathbb{Z}$.

Proof. Given $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ there exists a free resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

and hence, by Long Exact Sequence Lemma,

$$[M] = \sum_{i=0}^{n} (-1)^i [F_i] = \left(\sum_{i=0}^{n} (-1)^i \mathrm{rank} F_i \right) [R].$$

Therefore $G_0(R) = \mathbb{Z}[R]$ and, considering the rank function as in the previous example, we get $G_0(R) \simeq \mathbb{Z}$. □

Example 22. Let $(R, \mathfrak{m}, k)$ be an artinian local ring. Then $G_0(R) = \mathbb{Z}[k] \simeq \mathbb{Z}$.

Proof. By Corollary 24 we clearly get $G_0(R) = \mathbb{Z}[k]$. Using the length function, by Additive Map Lemma we get

$$\begin{aligned} G_0(R) &\longrightarrow \mathbb{Z} \\ [M] &\longmapsto \widetilde{\lambda}([M]) \end{aligned}$$

which is surjective since $\lambda(k) = 1$. Hence $G_0(R) \simeq \mathbb{Z}$ and moreover $[R] = \lambda(R)[k]$. □

Question (H. Dao)**.** *Let $(R, \mathfrak{m}, k)$ be a normal local ring, and assume k is an algebraically closed field of characteristic zero, $k \subseteq R$. Also, assume $G_0(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space. Does R have rational singularities?*

Theorem 27. *Let R be a noetherian ring and let $I \subseteq R$ be nilpotent. Then the map*

$$\begin{aligned} j : G_0(R/I) &\longrightarrow G_0(R) \\ [M] &\longmapsto [M] \end{aligned}$$

is an isomorphism.

Proof. First suppose we have shown the theorem when $I^2 = 0$. Then $G_0(R/I) \simeq G_0\left(R/I^2\right)$ since in R/I^2 clearly $I^2 = 0$. Similarly $G_0\left(R/I^2\right) \simeq G_0\left(R/I^4\right)$ and so on. For some $n \in \mathbb{N}$ we have $I^n = 0$, so that $R/I^n = R$ and hence we get the following chain of isomorphisms:

$$G_0(R/I) \xrightarrow{\simeq} G_0\left(R/I^2\right) \xrightarrow{\simeq} G_0\left(R/I^4\right) \xrightarrow{\simeq} \ldots\ldots \xrightarrow{\simeq} G_0(R).$$

So suppose $I^2 = 0$. Notice that under this assumption, given $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$, both IM and M/IM are R/I-modules. Consider the map $i : G_0(R) \to G_0(R/I)$ given by

$$i([M]) = [IM] + [M/IM].$$

Claim. $i : G_0(R) \to G_0(R/I)$ is well defined.

Proof of the Claim. On a basis $\{< M >: M \in \mathrm{Mod}^{\mathrm{fg}}(R)\}$ of $H_0(R)$ define

$$\begin{aligned} i' : H_0(R) &\longrightarrow G_0(R/I) \\ < M > &\longmapsto [IM] + [M/IM] \end{aligned}$$

and extend it to a group homomorphism. We need to prove that $i'(L(R)) = 0$. Let

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

be a short exact sequence of R-modules. Consider the following short exact sequences of R/I-modules:

$$0 \longrightarrow M_1 \cap IM \longrightarrow IM \longrightarrow IM_2 \longrightarrow 0 \tag{2.1}$$

$$0 \longrightarrow \tfrac{M_1 + IM}{IM} \longrightarrow \tfrac{M}{IM} \longrightarrow \tfrac{M_2}{IM} \longrightarrow 0 \tag{2.2}$$

$$0 \longrightarrow \tfrac{M_1 \cap IM}{IM_1} \longrightarrow \tfrac{M_1}{IM_1} \longrightarrow \tfrac{M_1}{M_1 \cap IM} \longrightarrow 0 \tag{2.3}$$

$$0 \longrightarrow IM_1 \longrightarrow M_1 \cap IM \longrightarrow \tfrac{M_1 \cap IM}{IM_1} \longrightarrow 0 \tag{2.4}$$

Then by (2.1) and (2.2) we get

$$i'(< M >) = [IM] + \left[\tfrac{M}{IM}\right] = ([M_1 \cap M] + [IM_2]) + \left(\left[\tfrac{M_1 + IM}{IM}\right] + \left[\tfrac{M_2}{IM_2}\right]\right) =$$

$$= i'(< M_2 >) + [M_1 \cap IM] + \left[\tfrac{M_1 + IM}{IM}\right].$$

Notice that $(M_1 + IM)/IM \simeq M_1/(M_1 \cap IM)$. Finally, by (2.3) and (2.4) we have

$$i'(< M >) = i'(< M_2 >) + [IM_1] + \left[\frac{M_1}{IM_1}\right] = i'(< M_1 >) + i'(< M_2 >).$$

So $i : G_0\,(R) \to G_0\,(R/I)$ is well defined.

Now let $[M] \in G_0\,(R/I)$ and consider the composition $(i \circ j)([M])$:

$$(i \circ j)([M]) = i([M]) = [IM] + [M/IM] = [M]$$

since $M \in \mathrm{Mod}^{\mathrm{fg}}(R/I)$ and hence $IM = 0$. Similarly, for $[M] \in G_0\,(R)$:

$$(j \circ i)([M]) = j\left([IM] + \left[\frac{M}{IM}\right]\right) = [IM] + \left[\frac{M}{IM}\right] = [M]$$

simply using the short exact sequence $0 \to IM \to M \to M/IM \to 0$. So j is an isomorphism. $\square$

Lemma 28 (Localization Lemma)**.** *Let R be a noetherian ring and W a multiplicatively closed set. Then there exists an exact sequence*

$$\bigoplus_{W \cap \mathfrak{p} \neq \emptyset} G_0\,(R/\mathfrak{p}) \xrightarrow{\alpha} G_0\,(R) \xrightarrow{\beta} G_0\,(R_W) \longrightarrow 0$$

where α and β are defined as follows:

$$\oplus[M(p)] \xmapsto{\alpha} \textstyle\sum[M(p)]$$

$$[M] \xmapsto{\beta} [M_W].$$

Proof. The first step is to show that

$$\mathrm{Im}(\alpha) = \langle [M] \mid M_W = 0 \rangle. \tag{2.5}$$

If M is a $R/\mathfrak{p}$ module and $\mathfrak{p} \cap W \neq \emptyset$, then clearly $M_W = 0$ since $\mathfrak{p} \subseteq \mathrm{ann}(M)$. Therefore

$$\mathrm{Im}(\alpha) \subseteq \langle [M] \mid M_W = 0 \rangle.$$

To see the other direction, suppose that $M_W = 0$. Since M is finitely generated, we know that

$$\mathrm{Supp}(M) = V(\mathrm{ann}(M)).$$

In a prime filtration of M, say,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_{i+1}/M_i \simeq R/\mathfrak{p}_i$, we have that $\mathfrak{p}_i \in \operatorname{Supp}(M)$. Thus the annihilator of M is contained in each $\mathfrak{p}_i$. But $M_W = 0$ implies there exists $w \in W$ such that $wM = 0$. In other words, $w \in \mathfrak{p}_i$ for all i and hence $\mathfrak{p}_i \cap W \neq \emptyset$ for all i as well. This forces $[R/\mathfrak{p}_i] \in \operatorname{Im}(\alpha)$ and by the Filtration lemma (Lemma 23) we have that $[M] \in \operatorname{Im}(\alpha)$. This proves (2.5).

Next we show that β is surjective. This is clear since every finitely generated R_W-module is of the form M_W for some finitely generated R-module M. (Note that β is well-defined as localization is flat.)

Observe that $\operatorname{Im}(\alpha) \subseteq \ker(\beta)$ since $M_W = 0$, that is, $\beta([M]) = 0$. Abusing notation, we now have the induced surjection

$$\frac{G_0\,(R)}{\operatorname{Im}(\alpha)} \xrightarrow{\ \beta\ } G_0\,(R_W)\,.$$

We want to show there exists a splitting γ, that is, an inverse to β. Define

$$\overline{\gamma} : H(R_W) \longrightarrow \frac{G_0\,(R)}{\operatorname{Im}(\alpha)}$$

by the following construction.

Let $\langle N\rangle \in H(R_W)$ and then choose an R-module M such that $M_W \simeq N$. Now define

$$\overline{\gamma}(\langle N\rangle) = [M] + \operatorname{Im}(\alpha).$$

To show this is well-defined, suppose $L_W \simeq N$. We need to prove that $[M]-[N]$ is an element of $\operatorname{Im}(\alpha)$. Since R is noetherian and all modules are finitely generated, we have that

$$(\mathrm{H}om_R(M, L))_W \simeq \mathrm{H}om_{R_W}(M_W, L_W). \tag{2.6}$$

As $M_W \simeq L_W$, choose some fixed R_W-isomorphism g. We can write $g = \frac{h}{w}$ where $h \in \mathrm{H}om_R(M, L)$. Replacing g by wg we get that

$$\frac{h}{1} \in \mathrm{H}om_{R_W}(M_W, L_W)$$

is an isomorphism. Consider the following exact sequences:

$$0 \longrightarrow \ker(h) \longrightarrow M \longrightarrow \operatorname{Im}(h) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im}(h) \longrightarrow L \longrightarrow \operatorname{coker}(h) \longrightarrow 0.$$

In $G_0\,(R)$, we have that

$$[M] - [L] = [\ker(h)] + [\operatorname{Im}(h)] - [\operatorname{Im}(h)] - [\operatorname{coker}(h)] = [\ker(h)] - [\operatorname{coker}(h)].$$

But, $(\ker(h))_W$ and $(\operatorname{coker}(h))_W$ are zero since $\frac{h}{1}$ is an isomorphism (use 2.5). It follows that $[M] - [L]$ is in the image of α and thus $\overline{\gamma}$ is a well-defined map.

Next, we would like to show that $L(R_W) \subseteq \ker(\overline{\gamma})$. Let

$$0 \longrightarrow N_1 \xrightarrow{f'} N \xrightarrow{g'} N_2 \longrightarrow 0 \tag{2.7}$$

be a short exact sequence of R_W-modules. We claim that

$$\overline{\gamma}(\langle N \rangle) = \overline{\gamma}(\langle N_1 \rangle) + \overline{\gamma}(\langle N_2 \rangle).$$

To see this, let M and M_2 be finitely generated R-modules such that $M_W = N_1$ and $(M_2)_W = N_2$. Using (2.6), choose $g : M \to M_2$ such that $\frac{g}{w} = g'$ where $w \in W$. Let $M_1 := \ker(g)$ and consider the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow \mathrm{Im}(g) \longrightarrow 0. \tag{2.8}$$

Since $(\mathrm{Im}(g))_W = N_2$, we have that

$$\begin{aligned} \overline{\gamma}(\langle N \rangle) &= [M] + \mathrm{Im}(\alpha); \\ \overline{\gamma}(\langle N \rangle_2) &= [\mathrm{Im}(g)] + \mathrm{Im}(\alpha). \end{aligned}$$

Thus, $\overline{\gamma}(\langle N \rangle) = [M_1] + \mathrm{Im}(\alpha)$ since the localization of (2.8) at W yields the short sequence (2.7). Therefore, $\overline{\gamma}(\langle N \rangle) - \overline{\gamma}(\langle N_1 \rangle) - \overline{\gamma}(\langle N_2 \rangle)$ is a coset of $[M] - [\mathrm{Im}(g)] - [M_1]$ in $G_0(R)/\mathrm{Im}(\alpha)$ and hence is 0 as we have an exact sequence. This shows that $L(R_W) \subseteq \ker(\overline{\gamma})$.

Finally, $\overline{\gamma}$ induces a homomorphism $\gamma : G_0(R_W) \to G_0(R)/\mathrm{Im}(\alpha)$. Let M be a finitely generated R-module. We have that $\gamma \cdot \beta$ is the identity, that is,

$$\begin{aligned} \gamma \cdot \beta([M] + \mathrm{Im}(\alpha)) &= \gamma([M_W]) \\ &= [M] + \mathrm{Im}(\alpha). \end{aligned}$$

Likewise, we have that $\beta \cdot \gamma$ is the identity. Let N be a finitely generated R_W-module and choose M finitely generated as an R-module such that $M_W = N$. Then,

$$\begin{aligned} \beta \cdot \gamma([N]) &= \beta([M] + \mathrm{Im}(\alpha)) \\ &= [M_W] \\ &= [N] \end{aligned}$$

□

Example 23. Let $R = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^5)$. One can prove that R is a 2-dimensional UFD. Then $G_0(R) = \mathbb{Z}[R] \simeq \mathbb{Z}$.

Proof. Notice that $(x) \subseteq R$ is prime. Consider the multiplicatively closed system $W = \{x^n : n \geq 1\}$, and notice that R_W is a 1-dimensional regular local ring, so that $G_0(R_W) \simeq \mathbb{Z}$. By Localization Lemma we have an exact sequence

$$\bigoplus_{W \cap \mathfrak{p} \neq \emptyset} G_0(R/\mathfrak{p}) \xrightarrow{\alpha} G_0(R) \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0$$

Notice that $\{\mathfrak{p} \in \mathrm{Spec}R : W \cap \mathfrak{p} \neq \emptyset\} = \{(x,y,z)R, (x)R\}$. Set $S := R/xR \simeq \mathbb{C}[[y,z]]/(y^3+z^5)$ so that the exact sequence becomes

$$G_0(\mathbb{C}) \oplus G_0(S) \xrightarrow{\alpha} G_0(R) \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0$$

By Corollary 24 we have $G_0(S) = \mathbb{Z}\{[S], [\mathbb{C}]\}$, since the only primes in S are the zero and the maximal ideal. Consider the following short exact sequence:

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow S \longrightarrow 0.$$

Then in the Grothendieck group $G_0(R)$ we have $[S] = 0$. Also notice that $x, y \in R$ form a regular sequence, therefore we get a long exact sequence

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \longrightarrow R/(x,y)R \longrightarrow 0.$$

So, inside $G_0(R)$:

$$\left[\frac{R}{(x,y)R}\right] = \left[\frac{\mathbb{C}[[z]]}{(z^5)}\right] = 5[\mathbb{C}] = 0$$

But this just means that $[\mathbb{C}]$ is torsion in $G_0(R)$. However, y, z also form a regular sequence in R, therefore

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \longrightarrow R/(y,z)R \longrightarrow 0.$$

is also exact. This means

$$\left[\frac{R}{(y,z)R}\right] = \left[\frac{\mathbb{C}[[x]]}{(x^2)}\right] = 2[\mathbb{C}] = 0$$

in $G_0(R)$, and hence $[\mathbb{C}] = 0$. Since $[\mathbb{C}] = [S] = 0$ in $G_0(R)$, we have that α is the zero map, and therefore

$$G_0(R) \simeq \mathbb{Z}.$$

□

Theorem 29. *Let R be a noetherian ring. Then the map*

$$\alpha : G_0(R) \longrightarrow G_0(R[x])$$

$$[M] \longmapsto [M \otimes_R R[x]]$$

is an isomorphism.

Proof. Let $M[x]$ denote $M \otimes_R R[x]$. Since $R[x]$ is free, in particular flat, the map α is well-defined. We will define a map $\beta : G_0(R[x]) \to G_0(R)$ with the following construction.

Let N be a finitely generated $R[x]$-module and consider the exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{x-1} N \longrightarrow C \longrightarrow 0$$

where K and C are the kernel and cokernel respectively. Notice in $G_0(R[x])$ that $[K]=[C]$ and that

$$(x-1)K=(x-1)C=0.$$

But $R \simeq R[x]/(x-1)$ and so K, C are finitely generated R-modules as well. Define

$$\beta([N])=[C]-[K]\in G_0(R)\,.$$

This is a well-defined map. To see this, define $\overline{\beta}(\langle N\rangle)=[C]-[K]$ and consider the following commutative diagram of $R[x]$-modules:

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r] \arrow[d, "x-1"] & N \arrow[r] \arrow[d, "x-1"] & N_2 \arrow[r] \arrow[d, "x-1"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N \arrow[r] & N_2 \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma, we have the long exact sequence

$$0 \longrightarrow K_1 \longrightarrow K \longrightarrow K_2 \longrightarrow C_1 \longrightarrow C \longrightarrow C_2 \longrightarrow 0$$

where K_i and C_i are the respective kernels and cokernels. Therefore we have that

$$[C]-[K]=[C_1]-[K_1]+[C_2]-[K_2],$$

giving the desired result of

$$\overline{\beta}(\langle N\rangle)=\overline{\beta}(\langle N_1\rangle)+\overline{\beta}(N_2).$$

Hence, β is well-defined.

Observe that the composition $\beta\circ\alpha$ is the identity. To see this, not that $x-1$ is a non-zero divisor on $M[x]$ and thus the sequence

$$0 \longrightarrow M[x] \xrightarrow{x-1} M[x] \longrightarrow M \longrightarrow 0 \tag{2.9}$$

is exact (this should be verified by the reader). Hence,

$$\beta([M[x]])=[M]-[0]=[M].$$

To finish the proof, it is enough to show that α is onto. To do this, we use Noetherian induction:

Assume not and choose and ideal I in R maximal such that

$$G_0(R/I) \xrightarrow{\alpha} G_0(R/I[x])$$

is not onto. We change notation by letting $R/I=R$.

Claim 1. The ring R is a domain.

To see this, suppose R is not a domain and let $\min(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Note that

$$\min(R[x]) = \{\mathfrak{p}_1 R[x], \ldots, \mathfrak{p}_s R[x]\}.$$

We thus have the following commutative diagram:

$$\begin{tikzcd}
\bigoplus G_0\left(R/\mathfrak{p}_i\right) \arrow[r] \arrow[d] & G_0\left(R\right) \arrow[r] \arrow[d, "\alpha"] & 0 \\
\bigoplus G_0\left(R/\mathfrak{p}_i[x]\right) \arrow[r] & G_0\left(R[x]\right) \arrow[r] & 0
\end{tikzcd}$$

By the corollary of the Filtration Lemma (page 32), the two horizontal maps are onto. Further, we have that the far left map is onto by induction. This forces α to be onto; a contradiction. Therefore, R is a domain.

Now consider the following sequence where $W = R \setminus \{0\}$:

$$\begin{tikzcd}
\bigoplus_{\substack{\mathfrak{p}\neq 0\\ \mathfrak{p}\in\operatorname{Spec}(R)}} G_0\left(\frac{R}{\mathfrak{p}}\right) \arrow[r] \arrow[rrr, bend right, "\gamma"] & \bigoplus G_0\left(\frac{R[x]}{\mathfrak{p}R[x]}\right) \arrow[r] & \bigoplus_{\substack{Q\cap W\neq\emptyset\\ Q\in\operatorname{Spec}(R[x])\\ Q\supseteq\mathfrak{p}R[x]}} G_0\left(\frac{R[x]}{Q}\right) \arrow[r] & G_0\left(R[x]\right).
\end{tikzcd}$$

The first map is onto by induction and the second is just a natural map and hence is also onto. The third map is defined by the Localization Lemma on page 36. As labeled above, let γ be the composition map.

Let k be the quotient field of R. We now have the following commutative diagram:

$$\begin{tikzcd}
 & \bigoplus G_0\left(\frac{R}{\mathfrak{p}}\right) \arrow[r] \arrow[d, "\gamma"] & G_0\left(R\right) \arrow[r] \arrow[d, "\alpha"] & G_0\left(k\right) \arrow[r] \arrow[d, "\alpha_k"] & 0 \\
0 \arrow[r] & K \arrow[r] & G_0\left(R[x]\right) \arrow[r] & G_0\left(k[x]\right) \arrow[r] & 0
\end{tikzcd}$$

where K is the respective kernel defined from the Localization Lemma. First notice that α_k is surjective since

$$G_0\left(k\right) = \mathbb{Z} = G_0\left(k[x]\right)$$

and α_k is defined by mapping $[k]$ to $[k[x]]$. Further, using the induction with the Filtration Lemma, we see that γ is onto the kernel K. Therefore, α is also onto by the snake lemma. $\square$

A helpful tool in the next theorem is the concept of the conductor.

Definition. Let R be a local domain and S be the integral closure of R. The conductor, denoted $\mathfrak{C}$, is the largest common ideal of both R and S.

Example 24. If $R = k[\![t^6, t^{10}, t^{15}]\!]$ then we have that $S = k[\![t]\!]$ is the integral closure. It is not difficult to see that $\mathfrak{C} = t^{30}R$.

Theorem 30 (Conjecture of Herzog)**.** *If $(R, \mathfrak{m}, k)$ is a one dimensional complete local domain with algebraically closed residue field, then $G_0(R) \simeq \mathbb{Z}$.*

Proof. By the Filtration Lemma, $G_0(R)$ is generated by $[R]$ and $[k]$. Further, we have that the rank function $G_0(R) \to \mathbb{Z}$ is onto and sends $[R]$ to 1. We need to prove that $[k] = 0$.

If $x \in R$ is non-zero, then $[R/xR] = 0$ and R/xR has a filtration of copies of k of $\lambda_R(R/xR)$. Therefore

$$\lambda_R(R/xR) \cdot [k] = 0$$

for any non-zero x in R. Let $S = \{\lambda(R/xR) \mid 0 \neq x \in \mathfrak{m}\}$. It is enough to show that the gcd of S is one.

Let V be the integral closure of R. Note that V is a one-dimensional (Cohen-Seidenberg), local (true sense R is complete; exercise) and V is integrally closed. Thus we have that V is a DVR with $\mathfrak{m}_V = (t)$.

As V is a finitely generated R-module, there exists a conductor $\mathfrak{C} \subseteq R$ such that $\mathfrak{C}V \subseteq R$. Pick any non-zero x in $\mathfrak{C}$. As tx is an element of $\mathfrak{C}$, we have that

$$\lambda_V(V/xV) + 1 = \lambda_V(V/txV).$$

However, as the residue fields of R and V are the same,

$$\begin{aligned}\lambda_V(V/xV) &= \lambda_R(V/xV)\\ &= \operatorname{rank}(V) \cdot \lambda_R(R/xR)\\ &= \lambda_R(R/xR).\end{aligned}$$

Therefore we have that $\lambda_R(R/xR) = \lambda_R(R/txR) - 1$. Thus we have that the gcd of S is one. □

1.1 Structure of One Dimensional Local Complete Domains

Assume that R contains its residue field k and that k is algebraically closed. Then $\overline{R} = k[\![t]\!]$ since $\overline{R}$ is a DVR as above and $\overline{R}/\mathfrak{m}_{\overline{R}} = k$; $k \subseteq \overline{R}$. Therefore, by Cohen's structure theorem, $\overline{R} = k[\![t]\!]$. Let

$$S = \min\{\lambda(\overline{R}/x\overline{R}) \mid x \in \mathfrak{m}\}.$$

Then there exists an element

$$t^s + \alpha_{s+1}t^{s+1} + \cdots \in R$$

and an element $u = 1 + \alpha_{s+1}t^s + \cdots$ such that $t^s u$ is the above element. Note that $t^s u \in R$, but u may not be in R.

If the characteristic of k is zero, then by Hensel's Lemma there exists $v \in \overline{R}$ such that $v^s = u$. Let $z = tv$. Then $k[\![t]\!] = k[\![z]\!]$. Now $z^s \in R$ and

$$R = k[\![z^s, \text{ higher order powers}]\!] \subseteq k[\![z]\!].$$

2 Class Groups

All along this section R will be an integrally closed noetherian domain.

Definition. Set $X^1(R) := \{\mathfrak{p} \in \mathrm{Spec}R : \mathrm{ht}\,\mathfrak{p} = 1\}$. Also set $X(R) =$ the free abelian group on generators $\mathfrak{p} \in X^1(R)$. More explicitly, if $D \in X(R)$ we can write

$$D = \sum_{\mathfrak{p} \in x^1(R)} n_{\mathfrak{p}} \mathfrak{p},$$

where all but finitely many among the $n_{\mathfrak{p}}$'s are zero. Elements in $X(R)$ are called divisors.

Notice that, since R is integrally closed, for all $\mathfrak{p} \in X^1(R)$ $R_{\mathfrak{p}}$ is a DVR (it is a 1-dimensional integrally closed local domain). If $\mathfrak{p}R_{\mathfrak{p}} = (t_{\mathfrak{p}})$, then every ideal is a power of the maximal ideal, i.e. $I = (t_{\mathfrak{p}}^n)$ for some $n \in \mathbb{N}$. By definition

$$(t_{\mathfrak{p}}^n)R_{\mathfrak{p}} \cap R = \mathfrak{p}^{(n)}$$

is the n-th symbolic power of $\mathfrak{p}$, and it is the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$.

Example 25. Lel k be a field, with $\mathrm{Char}k \neq 2$. Let

$$R := \frac{k[x, y, z]}{(x^2 - yz)}.$$

R is an integrally closed domain: R is Cohen-Macaulay, and hence it satisfies Serre's condition (S_2). Also the Jacobian ideal

$$J = (2x, y, z)$$

has height two, which means that $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in X^1(R)$ and hence R satisfies Serre's condition (R_1). Hence R is integrally closed.

Consider $\mathfrak{p} = (x, z)R$ a prime ideal in R, and notice that $\mathfrak{p} \in X^1(R)$. Then $\mathfrak{p}R_{\mathfrak{p}} = xR_{\mathfrak{p}}$, since in $R_{\mathfrak{p}}$ y is invertible and hence $z = -y^{-1}x^2$. Then $\mathfrak{p}^2 = (x^2, xz, z^2)R = (xz, z^2, yz)R$ and therefore

$$\mathfrak{p}^{(2)} = (xz, z^2, yz)R_{\mathfrak{p}} \cap R = (z)R_{\mathfrak{p}} \cap R = (z) \subsetneq \mathfrak{p}^2.$$

since y is again invertible in $R_{\mathfrak{p}}$.

Given a non-zero ideal $I \subseteq R$, set

$$\mathrm{div}\,(I) := \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(I)\mathfrak{p},$$

where

$$IR_{\mathfrak{p}} = \left(t^{v_{\mathfrak{p}}(I)}\right) R_{\mathfrak{p}} \quad \text{for } \mathfrak{p} \in X^1(R).$$

Since $I \neq (0)$ there are only finitely many minimal primes containing I, and hence only finitely many height one primes containing I. For the others, if $\mathfrak{p} \in X^1(R)$ and $I \not\subseteq \mathfrak{p}$, then $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$, and hence $v_{\mathfrak{p}}(I) = 0$ for such $\mathfrak{p}$. So $\mathrm{div}\,(I)$ is well defined, since we have just proved that the sum is finite.

Remark 24. If $\operatorname{ht} I \geqslant 2$, then $v_{\mathfrak{p}}(I) = 0$ for all $\mathfrak{p} \in X^1(R)$ and therefore $\operatorname{div}(I) = 0$.

Definition. Given $f = a/b \in K \smallsetminus \{0\}$, where K is the fraction field of R, define

$$\operatorname{div}(f) = \operatorname{div}(a) - \operatorname{div}(b).$$

Define the set of principal divisors

$$P(R) := \{\operatorname{div}(f) : f \in K \smallsetminus \{0\}\} \subseteq X(R).$$

It is a subgroup of $X(R)$ since

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g) \quad \text{and} \quad \operatorname{div}\left(f^{-1}\right) = -\operatorname{div}(f).$$

Finally define the Class Group of R as

$$\operatorname{Cl}(R) := X(R)/P(R).$$

The class of $\mathfrak{p} \in X(R)$ inside $\operatorname{Cl}(R)$ is denoted $[\mathfrak{p}]$.

Definition. Two divisors $D_1, D_2 \in X(R)$ are said to be linearly equivalent if $D_1 - D_2 \in P(R)$ (i.e. $[D_1] = [D_2]$). A divisor $D \in X(R)$ is said to be effective if

$$D = \sum_{\mathfrak{p} \in x^1(R)} n_{\mathfrak{p}} \mathfrak{p}.$$

and $n_{\mathfrak{p}} \geq 0$ for all $\mathfrak{p} \in X^1(R)$.

Lemma 31. *Let $D \in X(R)$. Then D is linearly equivalent to an effective divisor.*

Proof. Write $D = D^+ - D^-$, where D^+ and D^- are effective. Write

$$D^- = \sum_{i=1}^{k} n_i \mathfrak{p}_i$$

for some $\mathfrak{p}_i \in X^1(R)$, $n_i > 0$. Choose a non-zero x such that

$$x \in \mathfrak{p}_1^{(n_1)} \cap \ldots \cap \mathfrak{p}_k^{(n_k)} \neq (0),$$

then $E := D^+ - D^- + \operatorname{div}(x)$ is effective since $v_{\mathfrak{p}_i}(x) \geq n_i$ for all $i = 1, \ldots, k$ and clearly

$$D - E = \operatorname{div}(x) \in P(R).$$

□

Lemma 32. *Let $D \in X(R)$ be an effective divisor. Then $[D] = 0$ if and only if $D = \operatorname{div}(x)$ for some $x \in R$.*

Proof. If $D = \mathrm{div}\,(f) \in P(R)$, then by definition $[D] = 0$. Conversely assume $D = \mathrm{div}\,(a/b) = \mathrm{div}\,(a) - \mathrm{div}\,(b)$ for some $a, b \in R$, $b \neq 0$. Write

$$D = \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(a)\mathfrak{p} - \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(b)\mathfrak{p} = \sum_{\mathfrak{p} \in X^1(R)} (v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b))\mathfrak{p},$$

where $v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b) \geqslant 0$ since D is effective. By uniqueness of the minimal part of the primary decomposition we get

$$(a) = \bigcap_{\mathfrak{p} \in X^1(R)} \mathfrak{p}^{v_{\mathfrak{p}}(a)} \subseteq \bigcap_{\mathfrak{p} \in X^1(R)} \mathfrak{p}^{v_{\mathfrak{p}}(b)} = (b),$$

and therefore $a/b = x \in R$, so that $D = \mathrm{div}\,(a/b) = \mathrm{div}\,(x)$. □

Theorem 33. *R is UFD if and only if $\mathrm{Cl}\,(R) = 0$.*

Proof. R is UFD if and only if every height one prime is principal, if and only if $p = \mathrm{div}\,(x)$ for some $x \in R$. By the previous Lemma this is true if and only if $[p] = 0$ for all $\mathfrak{p} \in X(R)$, if and only if $\mathrm{Cl}\,(R) = 0$. □

Theorem 34 (Localization Lemma)**.** *Let $W \subseteq R$ be a multiplicatively closed set. Then there exists a short exact sequence*

$$0 \longrightarrow H \longrightarrow \mathrm{Cl}\,(R) \longrightarrow \mathrm{Cl}\,(R_W) \longrightarrow 0,$$

where is the subgroup $H =< [\mathfrak{p}] : \mathfrak{p} \in X^1(R), \mathfrak{p} \cap W \neq \emptyset >\subseteq \mathrm{Cl}\,(R)$.

Proof. First we define the map $\widetilde{\theta} : X(R) \longrightarrow X(R_W)$ as follows:

$$\widetilde{\theta}(\mathfrak{p}) = \begin{cases} \mathfrak{p} \ \text{ if } \ \cap W \neq (0) \\ \mathfrak{p}_W \ \text{ if } \ \mathfrak{p} \cap W = \emptyset \end{cases}$$

Notice that such a map is well defined since $X(R)$ is free and we can always define a map on a basis. Notice also that $\mathfrak{p}_W$ are height one primes in R_W.
Claim. $\widetilde{\theta}(P(R)) \subseteq P(R_W)$.

Proof of the Claim. By Lemma 31 it is enough to show it for effective divisors. Let $a \in R$, then

$$\mathrm{div}\,(a) = \sum_{\mathfrak{p} \cap W \neq \emptyset} v_{\mathfrak{p}}(a)\mathfrak{p} + \sum_{\mathfrak{q} \cap W = \emptyset} v_{\mathfrak{q}}(a)\mathfrak{q}.$$

Then

$$\widetilde{\theta}(\mathrm{div}\,(a)) = \sum_{\mathfrak{q} \cap W = \emptyset} v_{\mathfrak{q}}(a)\mathfrak{q}_W = \mathrm{div}\left(\frac{a}{1}\right).$$

This is because

$$(a) = \bigcap_{\mathfrak{p} \cap W \neq \emptyset} \mathfrak{p}^{v_{\mathfrak{p}}(a)} \cap \bigcap_{\mathfrak{q} \cap W = \emptyset} \mathfrak{q}^{v_{\mathfrak{q}}(a)}$$

and localizing

$$\left(\frac{a}{1}\right) = \bigcap_{\mathfrak{q}\cap W=\emptyset} \mathfrak{q}^{v_{\mathfrak{q}}(a)}.$$

$\square$

Also, $\widetilde{\theta}$ is surjective by the correspondence between primes in R and primes in R_W that don't intersect W. Therefore $\widetilde{\theta}$ induces a surjective map

$$\mathrm{Cl}\,(R) \longrightarrow \mathrm{Cl}\,(R_W) \xrightarrow{\theta} 0.$$

Clearly $H \subseteq \ker\theta$. Conversely, let $D \in \ker\theta$, we can assume that D is effective, so that

$$D = \sum_{\mathfrak{p}\cap W\neq\emptyset} d_{\mathfrak{p}}[\mathfrak{p}] + \sum_{\mathfrak{q}\cap W=\emptyset} d_{\mathfrak{q}}[\mathfrak{q}],$$

with $d_{\mathfrak{p}}, d_{\mathfrak{q}} \geqslant 0$, and $\sum_{\mathfrak{p}\cap W\neq\emptyset} d_{\mathfrak{p}}[\mathfrak{p}] \in H \subseteq \ker\theta$. Also

$$\theta\left(\sum_{\mathfrak{q}\cap W=\emptyset} d_{\mathfrak{q}}[\mathfrak{q}]\right) = \sum_{\mathfrak{q}\cap W=\emptyset} d_{\mathfrak{q}}[\mathfrak{q}_w] = 0$$

inside $\mathrm{Cl}\,(R_W)$. This means that there exists $\frac{a}{w} \in R_W$ such that

$$\mathrm{div}\,(a/w) = \sum_{\mathfrak{q}\cap W=\emptyset} d_{\mathfrak{q}}[\mathfrak{q}_w].$$

Since w is a unit in R_W we have $\mathrm{div}\,(a/w) = \mathrm{div}\,(a/1)$, so we can assume

$$\mathrm{div}\,(a/1) = \sum_{\mathfrak{q}\cap W=\emptyset} d_{\mathfrak{q}}[\mathfrak{q}_w].$$

Taking a primary decomposition of (a), we have that the part with $\{\mathfrak{p} : \mathfrak{p}\cap W = \emptyset\}$ has to be the same as for D, so that

$$\mathrm{div}\,(a) = \sum_{\mathfrak{p}\cap W\neq\emptyset} e_{\mathfrak{p}}\mathfrak{p} + \sum_{\mathfrak{q}\cap W=\emptyset} d_{\mathfrak{q}}[\mathfrak{q}].$$

Clearly $D \cong \mathrm{div}\,(a) \pmod{H}$, so that $[D] \cong 0 \pmod{H}$ in $\mathrm{Cl}\,(R)$ and therefore $[D] \in H$. $\square$

Example 26. Let k be a field with $\mathrm{Char} k \neq 2$. Let

$$R = \frac{k[x,y,z]}{(x^2 - yz)}$$

and consider the multiplicatively closed set $W = \{z^n\}$. Then

$$R[z^{-1}] \simeq \frac{k[x,y,z,z^{-1}]}{(x^2 - yz)} \simeq \frac{k[x,y,z,z^{-1}]}{(x^2z^{-1} - y)} \simeq k[x,z,z^{-1}],$$

which is a UFD, and hence $\mathrm{Cl}\,(R)_W = 0$. Using Localization Lemma for Class Groups we have that

$$\mathrm{Cl}\,(R) \simeq < [\mathfrak{p}] : \mathfrak{p} \cap \{z^n\} \neq \emptyset, \mathfrak{p} \in X^1(R) > .$$

Notice that $\sqrt{zR} = (x, z) =: \mathfrak{p}$ and $\mathrm{ht}\,\mathfrak{p} = 1$, so that it is the only height one prime intersecting W. Hence $\mathrm{Cl}\,(R) \simeq \mathbb{Z}[\mathfrak{p}]$. We have already noticed that

$$\mathfrak{p}^{(2)} = (z)$$

hence $\mathrm{div}\,(z) = 2\mathfrak{p}$, which means $2[\mathfrak{p}] = 0$ in $\mathrm{Cl}\,(R)$. We have two cases:

$$\mathrm{Cl}\,(R) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{cases}$$

But R is not a UFD because $\mathfrak{p}$ is height one but not principal, therefore $\mathrm{Cl}\,(R) \neq 0$, that is

$$\mathrm{Cl}\,(R) \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Theorem 35. *Let R be an integrally closed noetherian domain. Then*

(1) $R[T_1, \ldots, T_n]$ is an integrally closed noetherian domain.

(2) $\mathrm{Cl}\,(R) \simeq \mathrm{Cl}\,(R[T_1, \ldots, T_n])$.

Proof. We prove only (2). By induction is enough to show the case $n = 1$. Recall that if $\mathfrak{p} \in X^1(R)$, then $\mathfrak{p}[T] = \mathfrak{p} \otimes_R R[T] \in X^1(R[T])$. Also, if $a \in R$ and

$$(a) = \mathfrak{p}_1^{(n_1)} \cap \ldots \cap \mathfrak{p}_k^{(n_k)}$$

is a primary decomposition, then

$$aR[T] = \mathfrak{p}_1[T]^{(n_1)} \cap \ldots \cap \mathfrak{p}_k[T]^{(n_k)}. \tag{2.10}$$

Let $W = R \setminus \{0\}$, then $(R[T])_W = R_W[T] = K[T]$, where $K = Q(R)$ is the quotient field of R. This is UFD since it is a PID, hence $\mathrm{Cl}\,(()\,R[T])_W) = 0$. Using Localization Lemma for Class Groups we have

$$\mathrm{Cl}\,(R[T]) \simeq H = < [Q] : Q \cap W \neq \emptyset, Q \in X^1(R[T]) > .$$

Note that, since $Q \cap R \neq (0)$, then $\mathfrak{q} = Q \cap R$ is a height one prime in R. Therefore $Q \supseteq \mathfrak{q}[T]$. But $\mathrm{ht}\,Q = 1$, hence $Q = \mathfrak{q}[T]$. Hence

$$X(R) \longrightarrow H \longrightarrow 0$$

$$\mathfrak{q} \longmapsto [\mathfrak{q}[T]]$$

By (2.10) the kernel is exactly $P(R)$, hence $H \simeq \mathrm{Cl}\,(R)$. This is because principal divisors in R correspond to principal divisors in $R[T]$. $\square$

Remark 25. In general it is not true that $\mathrm{Cl}(R) \simeq \mathrm{Cl}(R[\![T]\!])$. If this is the case R is said to have discrete divisor class group. Notice that we have always

$$\mathrm{Cl}(R) \hookrightarrow \mathrm{Cl}(R[\![T]\!])$$

Theorem 36 (Danilov)**.** *If R satisfies Serre's conditions (S_3) and (R_2) then* $\mathrm{Cl}(R) \simeq \mathrm{Cl}(R[\![T]\!])$.

Remark 26. Similarly we always have $\mathrm{Cl}(R) \hookrightarrow \mathrm{Cl}\left(\widehat{R}\right)$.

Theorem 37 (Flenner)**.** *Let R be an integrally closed standard graded domain, say $R = k[R_1] = \bigoplus_{i\geqslant 0} R_i$ where k is a field. Set $\mathfrak{m} = \bigoplus_{i\geqslant 1} R_i$ and assume R satisfies Serre's condition (R_2). Then*

$$\mathrm{Cl}(R) \simeq \mathrm{Cl}(R_{\mathfrak{m}}) \simeq \mathrm{Cl}\left(\widehat{R_{\mathfrak{m}}}\right).$$

Example 27. Let

$$R = \left(\frac{\mathbb{C}[x,y,z]}{(x^2+y^3+z^7)}\right)_{(x,y,z)}.$$

Then R is UFD, therefore $\mathrm{Cl}(R) = 0$, but $\mathrm{Cl}(R[\![T]\!]) \neq 0$ and $\mathrm{Cl}\left(\widehat{R}\right) \neq 0$.

There is a relation between the Class Group and the Grothendieck Group. First we need the following definition.

Definition. Let R be a noetherian domain. The reduced Grothendieck Group $\widetilde{G_0}(R)$ is the subgroup of $G_0(R)$ which is the kernel of the map

$$G_0(R) \longrightarrow G_0(K),$$

where $K = Q(R)$ is the fraction field of R. Notice that, by Localization Lemma, $\widetilde{G_0}(R)$ is generated by $[R/\mathfrak{p}]$, where $\mathfrak{p} \neq 0$ is prime.

Theorem 38. *Let R be a integrally closed noetherian domain. Let H be the subgroup of $G_0(R)$ generated by $[R/\mathfrak{p}]$, with $\operatorname{ht}\mathfrak{p} = 2$. Then there is a short exact sequence*

$$0 \longrightarrow H \longrightarrow \widetilde{G_0}(R) \longrightarrow \mathrm{Cl}(R) \longrightarrow 0.$$

The proof of the theorem is postponed.

3 Divisors attached to Modules

Throughout this section let R be an integrally closed noetherian domain. The goal is to construct a map

$$c : G_0(R) \longrightarrow \mathrm{Cl}(R)$$

$$[M] \longmapsto c([M])$$

We will start the construction restricting to torsion modules.

Definition. Let R be a domain. An R-module T is said to be torsion if there exists $x \in R$, $x \neq 0$ such that $xT = 0$.

Let T be a torsion R-module. Define

$$c(T) = \sum_{\mathfrak{p} \in X^1(R)} \lambda(T_\mathfrak{p})[\mathfrak{p}] \in \mathrm{Cl}\,(R)\,.$$

Since T is torsion we have $\mathrm{ht}\, annT \geq 1$. Also, since $\mathrm{Supp}T = V(annT)$, there exist only finitely many $\mathfrak{p} \in X^1(R)$ such that $\ in\mathrm{Supp}T$, namely

$$\mathrm{Supp}T \cap X^1(R) = X^1(R) \cap \mathrm{Min}(annT).$$

For each of these we get $\lambda(T_\mathfrak{p}) < \infty$ since $\sqrt{(annT)_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$, which means that $\mathrm{Supp}T_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$ is just the maximal ideal, hence $T_\mathfrak{p}$ has finite length.

Remark 27. If we have a short exact sequence of torsion modules

$$0 \longrightarrow T_1 \longrightarrow T \longrightarrow T_2 \longrightarrow 0,$$

then $c(T) = c(T_1) + c(T_2)$ since localization is flat and length is additive.

Let us now go back to the general case. Let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$, and recall that $\mathrm{rank}(M) = \dim_K M \otimes_R K$, where $K = R_{(0)}$ is the fraction field of R.

Remark 28. T is torsion of and only if $\mathrm{rank}(T) = 0$.

Suppose $\mathrm{rank}(M) = r$, then there is a K-vector space isomorphism

$$\widetilde{\alpha} : R^r \otimes_R K \longrightarrow M \otimes_R K.$$

Since R^r is finitely presented, we have

$$\mathrm{H}om_{R_{(0)}}(R^r_{(0)}, M_{(0)}) \simeq (\mathrm{H}om_R(R^r, M))_{(0)}\,,$$

which means that there exists $\alpha : R^r \to M$ such that $\alpha_{(0)} = \widetilde{\alpha}$. We have an exact sequence

$$0 \longrightarrow \ker\alpha \longrightarrow R^r \longrightarrow M \longrightarrow \mathrm{coker}\alpha \longrightarrow 0,$$

and also $\ker_\alpha$ and $\mathrm{coker}\alpha$ are torsion modules because $\alpha_{(0)} = \widetilde{\alpha}$, which is an isomorphism. Hence $\ker\alpha \otimes 1 = 0 = \mathrm{coker}\alpha \otimes 1$. But $\ker\alpha \subseteq R^r$, and a submodule of a free module cannot be torsion unless it is zero. Therefore we have the following short exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow T \longrightarrow 0,$$

where F is free and $T := \mathrm{coker}\alpha$ is torsion. Define $c(M) := c(T)$.

Claim. c is well defined.

Proof of the Claim. Suppose we have

$$0 \longrightarrow F \longrightarrow M \longrightarrow T \longrightarrow 0$$

and

$$0 \longrightarrow G \longrightarrow M \longrightarrow T' \longrightarrow 0.$$

We need to prove that $c(T) = c(T')$. First of all we can reduce to the case $F \subseteq G$. In fact, notice that

$$F \otimes_R K \simeq G \otimes_R K \simeq M \otimes_R K \simeq K^r.$$

We can think of F and G inside K^r (they are not K-vector subspaces). Then there exists $x \in R$ such that $xF \subseteq G$. However, consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \ker\theta \arrow[d] & \\
0 \arrow[r] & xF \arrow[r] \arrow[d, "i"] & M \arrow[r] \arrow[d, "\mathrm{id_M}"] & T'' \arrow[r] \arrow[d, dashed, "\theta"] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
 & \dfrac{F}{xF} & 0 & \mathrm{coker}\theta &
\end{tikzcd}$$

By the Snake Lemma we get $\ker\theta \simeq F/xF$, and hence we have a short exact sequence

$$0 \longrightarrow \tfrac{F}{xF} \longrightarrow T'' \longrightarrow T \longrightarrow 0$$

These are all torsion modules, and we have already proved that for torsion modules c is additive. Hence

$$c(T'') = c(T) + c(F/xF) = c(T) + rc(R/xR).$$

But

$$c(\frac{R}{xR}) = \sum_{\mathfrak{p} \in X^1(R)} \lambda\left(\left(\frac{R}{xR}\right)_{\mathfrak{p}}\right)[\mathfrak{p}] = [\mathrm{div}\,(x)] = 0 \ \text{ in } \mathrm{Cl}\,(R)\,.$$

Therefore $c(T'') = c(T)$, and hence without loss of generality we can assume $F \subseteq G$.

Now consider

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker\delta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d, "i"'] & M \arrow[r] \arrow[d, "\mathrm{id_M}"'] & T \arrow[r] \arrow[d, dashed, "\delta"'] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \dfrac{G}{F} \arrow[r] & 0 \arrow[r] & \mathrm{coker}\delta \arrow[r] & 0
\end{tikzcd}$$

so that, again by Snake Lemma, we get a short exact sequence

$$0 \longrightarrow \tfrac{G}{F} \longrightarrow T \longrightarrow T' \longrightarrow 0,$$

and therefore

$$c(T) = c(T') + c(G/F).$$

We want to prove that $c(G/F) = 0$. We have

$$0 \longrightarrow F \simeq R^r \xrightarrow{A} G \simeq R^r \longrightarrow \frac{G}{F} \longrightarrow 0,$$

and we need to prove that there exists $y \in R$ such that

$$\sum_{\mathfrak{p}\in X^1(R)} \lambda\left(\left(\frac{G}{F}\right)_{\mathfrak{p}}\right)\mathfrak{p} = \operatorname{div}(y).$$

Take $y = \det A$. Let $\mathfrak{p} \in X^1(R)$, then localizing we get

$$0 \longrightarrow R_{\mathfrak{p}}^r \xrightarrow{A_{\mathfrak{p}}} R_{\mathfrak{p}}^r \longrightarrow \frac{G_{\mathfrak{p}}}{F_{\mathfrak{p}}} \longrightarrow 0.$$

Since $\mathfrak{p}$ is a height one prime, $R_{\mathfrak{p}}$ is a DVR, and in particular a PID. Then we can use the fundamental theorem for PID, for which we can diagonalize $A_{\mathfrak{p}}$ changing basis. So assume

$$A_{\mathfrak{p}} = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_r \end{pmatrix}$$

is diagonal, with $d_i \neq 0$ for all i since $A_{\mathfrak{p}}$ is an injection. Then

$$\frac{G_{\mathfrak{p}}}{F_{\mathfrak{p}}} \simeq \left(\frac{R}{(d_1)}\right)_{\mathfrak{p}} \oplus \left(\frac{R}{(d_2)}\right)_{\mathfrak{p}} \oplus \dots \oplus \left(\frac{R}{(d_r)}\right)_{\mathfrak{p}}$$

and therefore

$$\lambda\left(\frac{G_{\mathfrak{p}}}{F_{\mathfrak{p}}}\right) = \sum_{i=1}^{r} \lambda\left(\left(\frac{R}{(d_i)}\right)_{\mathfrak{p}}\right).$$

Finally

$$\sum_{i=1}^{r} \lambda\left(\left(\frac{R}{(d_i)}\right)_{\mathfrak{p}}\right) = \lambda\left(\frac{R_{\mathfrak{p}}}{(d_1 \cdot \ldots \cdot d_r)_{\mathfrak{p}}}\right) = \lambda\left(\left(\frac{R}{(\det A)}\right)_{\mathfrak{p}}\right),$$

so that

$$c\left(\frac{G}{F}\right) = [\operatorname{div}(\det a)] = 0 \text{ in } \operatorname{Cl}(R).$$

□

We want to prove now that c induces a map on $G_0(R)$. Suppose we have a short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2 \longrightarrow 0,$$

we want to show that $c(M) = C(M_1) + c(M_2)$. Suppose $\operatorname{rank}(M_1) = r_1$ and $\operatorname{rank}(M_2) = r_2$ and choose free modules $R^{r_1} \subseteq M_1$ and $R^{r_2} \subseteq M_2$ with bases $e_1, \ldots, e_{r_1}$ and $f_1, \ldots, f_{r_2}$ respectively. Set

$$u_i = \alpha(e_j) \in M \text{ for } i = 1, \ldots, r_1 \text{ and } v_j = \beta^{-1}(f_j) \in M \text{ for } j = 1, \ldots, r_2.$$

Claim. $R\{u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}\}$ is a free R-module of rank $r_1 + r_2$.

Proof of the Claim. Assume not, so that there exist $s_1, \ldots, s_{r_2}, t_1, \ldots, t_{r_2} \in R$ not all zero such that

$$s_1 u_1 + \ldots + s_{r_1} u_{r_1} + t_1 v_1 + \ldots + t_{r_2} v_{r_2} = 0.$$

Apply β, using the fact that $\beta(u_i) = \beta(\alpha(e_i)) = 0$ for all $i = 1, \ldots, r_1$:

$$0 = \sum_{j=1}^{r_2} t_j \beta(v_j) = \sum_{j=1}^{r_2} t_j f_j,$$

which implies $t_1 = \ldots = t_{r_2} = 0$ since $\{f_1, \ldots, f_{r_2}$ is a basis of R^{r_2}. Then we have

$$\sum_{i=1}^{r_1} s_i u_i = \sum_{i=1}^{r_1} \alpha(e_i) = 0,$$

which implies $u_1 = \ldots = u_{r_1} = 0$ since $\{e_1, \ldots, e_{r_1}\}$ is a basis of R^{r_1} and α is injective. This proves the Claim. □

Finally we have an exact diagram

$$\begin{tikzcd}
0 \arrow[r] & R^{r_1} \arrow[r] \arrow[d, hook] & R^{r_1+r_2} \arrow[r] \arrow[d, hook] & R^{r_2} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_1 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which commutes by construction. The short exact sequence

$$0 \longrightarrow T_1 \longrightarrow T \longrightarrow T_2 \longrightarrow 0$$

exists by Snake Lemma, and T_1, T and T_2 are all torsion modules, again by construction. But for torsion modules we know that c is additive, hence

$$c(M) = c(T) = c(T_1) + c(T_2) = c(M_1) + c(M_2).$$

Summarizing we have the following theorem.

Theorem 39. *There is a surjective group homomorphism*

$$c : \widetilde{G_0}\,(R) \longrightarrow \mathrm{Cl}\,(R) \longrightarrow 0$$

Proof. We already proved that there is a map

$$c : H(R) \longrightarrow \mathrm{Cl}\,(R)$$

that preserves short exact sequences, i.e. $c(L(R)) = 0$. This induces a group homomorphism

$$c : G_0\,(R) \longrightarrow \mathrm{Cl}\,(R)\,.$$

To prove that it is surjective let $[\mathfrak{p}] \in \mathrm{Cl}\,(R)$ and consider $R/\mathfrak{p}$, which is a torsion R-module. Then

$$c\left(\left[\frac{R}{\mathfrak{p}}\right]\right) = \sum_{\mathfrak{q} \in X^1(R)} \lambda\left(\left(\frac{R}{\mathfrak{p}}\right)_{\mathfrak{q}}\right)[\mathfrak{q}] = [\mathfrak{p}].$$

Finally we can consider the restriction

$$c : \widetilde{G_0}\,(R) \longrightarrow \mathrm{Cl}\,(R)\,,$$

which is still surjective since for all $\mathfrak{p} \in X^1(R)$ we have $[R/\mathfrak{p}] \in \widetilde{G_0}\,(R)$. $\square$

Proposition 40. *Let* $c : \widetilde{G_0}(R) \to \mathrm{Cl}(R)$ *be as above. Then*

$$\ker c = H =< [R/\mathfrak{p}] : \mathrm{ht}\, \mathfrak{p} \geq 2 >,$$

so that we have a short exact sequence

$$0 \longrightarrow H \longrightarrow \widetilde{G_0}(R) \longrightarrow \mathrm{Cl}(R) \longrightarrow 0.$$

Proof. First notice that $H \subseteq \ker c$ since if $[R/\mathfrak{p}] \in H$, then $R/\mathfrak{p}$ is torsion and hence

$$c([R/\mathfrak{p}]) = \sum_{\mathfrak{q} \in X^1(R)} \lambda\left((R/\mathfrak{p})_{\mathfrak{q}}\right)[\mathfrak{q}] = 0$$

since, being $\mathrm{ht}\, \mathfrak{p} \geq 2$, $\mathfrak{p}$ cannot be contained in any height one prime. To prove the converse we want to find a left inverse to c. First let use define a map on the free abelian group $X(R)$ as follows:

$$\beta : X(R) \longrightarrow \widetilde{G_0}(R)/H$$

$$\mathfrak{p} \longmapsto [R/\mathfrak{p}] + H$$

Notice that β is onto since by the localization lemma we have

$$\widetilde{G_0}(R) =< [R/\mathfrak{q}] : \mathrm{ht}\, \mathfrak{q} \geq 1 >$$

and H already involves all primes of height at least two, while $\beta(X(R))$ involves all primes of height one. We now want to show that $P(R) \subseteq \ker \beta$. Let $a \in R$, $a \neq 0$, and write

$$(a) = \bigcap_{\mathfrak{p} \in X^1(R)} \mathfrak{p}^{(n_{\mathfrak{p}})},$$

so that

$$\mathrm{div}\,(a) = \sum_{\mathfrak{p} \in X^1(R)} n_{\mathfrak{p}}[\mathfrak{p}] \in X(R).$$

Applying β we get

$$\beta(\mathrm{div}\,(a)) = \sum_{\mathfrak{p} \in X^1(R)} n_{\mathfrak{p}}[R/\mathfrak{p}] + H \in \widetilde{G_0}(R).$$

Consider the following short exact sequence

$$0 \longrightarrow \frac{R}{(a)} \longrightarrow \bigoplus_{\mathfrak{p} \in X^1(R)} \frac{R}{\mathfrak{p}^{(n_{\mathfrak{p}})}} \longrightarrow T \longrightarrow 0, \tag{2.11}$$

where T is the cokernel of the first map. Also, notice that we have the following short exact sequence of R-modules

$$0 \longrightarrow R \xrightarrow{\cdot a} R \longrightarrow R/(a) \longrightarrow 0,$$

so that $[R/(a)] = 0$ in $G_0(R)$. Therefore

$$\left[\bigoplus_{\mathfrak{p}\in X^1(R)} \frac{R}{\mathfrak{p}^{n_\mathfrak{p}}}\right] = \left[\frac{R}{(a)}\right] + [T] = [T].$$

Claim. $\mathrm{ht}(annT) \geq 2$.

Proof of the Claim. Let $\mathfrak{q} \in X^1(R)$. If $a \in \mathfrak{q}$, localizing the short exact sequence (2.11) at $\mathfrak{q}$ we get

$$0 \longrightarrow \left(\frac{R}{(a)}\right)_\mathfrak{q} = \frac{R}{\mathfrak{q}^{(n_\mathfrak{q})}} \longrightarrow \left(\bigoplus_{\mathfrak{p}\in X^1(R)} \frac{R}{\mathfrak{p}^{(n_\mathfrak{p})}}\right)_\mathfrak{q} = \frac{R}{\mathfrak{q}^{(n_\mathfrak{q})}} \longrightarrow T_\mathfrak{q} \longrightarrow 0,$$

so that $T_\mathfrak{q} = 0$. Also, if $a \notin \mathfrak{q}$, then both $(R/(a))_\mathfrak{q}$ and $\left(\bigoplus_{\mathfrak{p}\in X^1(R)} R/\mathfrak{p}^{(n_\mathfrak{p})}\right)_\mathfrak{q}$ are zero, so that $T_\mathfrak{q} = 0$ again. This proves the Claim. □

By the Claim we have that

$$\left[\bigoplus_{\mathfrak{p}\in X^1(R)} \frac{R}{\mathfrak{p}^{(n_\mathfrak{p})}}\right] = [T] \in H.$$

Finally, one can prove that

$$\left[\bigoplus_{\mathfrak{p}\in X^1(R)} \frac{R}{\mathfrak{p}^{(n_\mathfrak{p})}}\right] - \sum_{\mathfrak{p}\in X^1(R)} n_\mathfrak{p} \left[\frac{R}{\mathfrak{p}}\right] \in H,$$

so that $P(R) \subseteq \ker\beta$, and hence we get an induced map

$$\beta : \mathrm{Cl}(R) \longrightarrow \widetilde{G_0}(R)/H.$$

To finish the proof we need to show that β is a left inverse for c, and it is enough to check it on the generators of $\widetilde{G_0}(R)/H$, i.e. $[R/\mathfrak{p}] + H$ with $\mathfrak{p} \in X^1(R)$. Notice that

$$c\left(\left[\frac{R}{\mathfrak{p}}\right] + H\right) = \sum_{\mathfrak{q}\in X^1(R)} \lambda\left(\left(\frac{R}{\mathfrak{p}}\right)_\mathfrak{q}\right)[\mathfrak{q}] = \left[\frac{R}{\mathfrak{p}}\right],$$

so that

$$\beta\circ c\left(\left[\frac{R}{\mathfrak{p}}\right] + H\right) = \beta\left(\left[\frac{R}{\mathfrak{p}}\right]\right) = \left[\frac{R}{\mathfrak{p}}\right] + H,$$

that is $\beta\circ c = \mathrm{id}_{\widetilde{G_0}(R)/H}$, and c is injective. This implies $\ker c = H$. □

Corollary 41. *Let R be a integrally closed noetherian domain such that every $\mathfrak{p} \in X^1(R)$ has finite free resolution. Then R is a UFD.*

Proof. It is enough to show that $\mathrm{Cl}\,(R) = 0$. Let $[\mathfrak{p}] \in \mathrm{Cl}\,(R)$, then by assumption there is a finite free resolution

$$0 \longrightarrow F_n \qquad \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \tfrac{R}{\mathfrak{p}} \longrightarrow 0$$

with the F_i's finitely generated free R-modules. In $\widetilde{G_0}\,(R)$ we get

$$\left[\frac{R}{\mathfrak{p}}\right] = \sum_{i=0}^{n}(-1)^i[F_i],$$

and applying c we get

$$c\left(\left[\frac{R}{\mathfrak{p}}\right]\right) = [p] = \sum_{i=0}^{n}(-1)^i c([F_i]) = 0$$

since c is zero on free modules. Therefore $\mathrm{Cl}\,(R) = 0$ and R is a UFD. □

We now state, without proving it, Eagon's Theorem.

Theorem 42 (Eagon)**.** *Let R be a noetherian ring and let F be the free abelian group on $\{\langle R\mathfrak{p}\rangle : \mathfrak{p} \in \mathrm{Spec}R\}$. Let $W \subseteq F$ be the submodule generated by all*

$$\sum_{\mathfrak{p}\in\Lambda} n_{\mathfrak{p}}\langle R/\mathfrak{p}\rangle,$$

where for $\mathfrak{q} \in \mathrm{Spec}R$ and $x \notin \mathfrak{q}$ there exists a prime filtration of $R/(\mathfrak{q}, x)$ that has exactly $n_{\mathfrak{p}}$ copies of $R/\mathfrak{p}$, for $\mathfrak{p} \in \Lambda$. Then

$$G_0\,(R) \simeq F/W.$$

Remark 29. For $\mathfrak{q} \in \mathit{spec}R$ and $x \notin \mathfrak{q}$ we always have a short exact sequence

$$0 \longrightarrow \frac{R}{\mathfrak{q}} \xrightarrow{\cdot x} \frac{R}{\mathfrak{q}} \longrightarrow \tfrac{R}{(\mathfrak{q},x)} \longrightarrow 0,$$

therefore $[R/(\mathfrak{q}, x)] = 0$ in $G_0\,(R)$ and we always have a subjective map

$$F/W \to G_0\,(R) \to 0.$$

4 Construction and Properties of $K_0\,(R)$

Definition. An R-module P is said to be projective if there exist an R-module Q and a free R-module F such that

$$P \oplus Q \simeq F.$$

Remark 30. By symmetry, the module Q in the above definition is also projective.

Remark 31. Free R-modules are projective.

Remark 32. If $(R, \mathfrak{m})$ is a local noetherian ring, then every projective R-module P is free. With the assumption that P is finitely generated this is an easy exercise, using Nakayama's Lemma. The result for non-finitely generated modules was proved by Kaplansky.

Remark 33. If R is a noetherian ring and P is Projective, then for all $\mathfrak{q} \in \mathrm{Spec}R$ $P_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module. This is because localization commutes with direct sums, and because projective modules over a local ring are free by Remark 31. If P is finitely generated, then the converse holds, i.e. if $P_{\mathfrak{q}}$ is free for all $\mathfrak{q} \in \mathrm{Spec}R$, then P is projective.

Remark 34. Let $W \subseteq R$ be a multiplicatively closed set. Because localization commutes with direct sums we have that if P is a projective R-module, then P_W is a projective R_W-module.

Remark 35. Let P be a R-module. Then P is projective if and only if whenever we have a short exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow P \longrightarrow 0$$

then it splits, i.e. $M \simeq P \oplus K$.

Definition. Let R be a noetherian ring. In analogy with $G_0(R)$ we define

$$K_0(R) := \frac{\mathbb{Z}\{\text{finitely generated projective } R\text{-modules}\}}{\mathbb{Z}\{< P \oplus Q > - < P > - < Q >\}}$$

since by Remark 35 every short exact sequence of Projective modules splits.

Remark 36. The case in which $(R, \mathfrak{m})$ is local is not interesting since every Projective module is free by Remark 31.

Remark 37. Clearly there is a map

$$\begin{tikzcd} K_0(R) \arrow[r] & G_0(R) \arrow[r, dashed, "\mathrm{rank}, e, \lambda"] & \mathbb{Z} \\ {[P]} \arrow[r, mapsto] & {[P]} & \end{tikzcd}$$

so that we can apply the same results concerning the existence of functions

$$\mathrm{rank}, e, \lambda : K_0(R) \longrightarrow \mathbb{Z}$$

as we did for $G_0(R)$.

Theorem 43. *Let R be a noetherian ring and let P, Q be finitely generated projective R-modules. Then $[P] = [Q]$ in $K_0(R)$ if and only if there exists a finitely generated free R-module F such that*

$$P \oplus F \simeq Q \oplus F.$$

Proof. Clearly if $P \oplus F \simeq Q \oplus F$ for some finitely generated free R-module F, then $[P] = [Q]$ in $K_0(R)$. Conversely assume $[P] = [Q]$. Then there exist short exact sequences

$$0 \longrightarrow P_i \longrightarrow P_i \oplus Q_i \longrightarrow Q_i \longrightarrow 0$$

such that

$$\langle P\rangle - \langle Q\rangle = \sum_{a_i>0} a_i(\langle P_i \oplus Q_i\rangle - \langle P_i\rangle - \langle Q_i\rangle) - \sum_{b_j>0} b_j(\langle P_j \oplus Q_j\rangle - \langle P_j\rangle - \langle Q_j\rangle),$$

where we rearrange the sum in order to have positive coefficients. then we can rewrite

$$\langle P\rangle + \sum_{a_i>0} a_i\langle P_i \oplus Q_i\rangle + \sum_{b_j>0} b_j(\langle P_j\rangle + \langle Q_j\rangle) = \langle Q\rangle + \sum_{a_i>0} a_i(\langle P_i\rangle + \langle Q_i\rangle) + \sum_{b_j>0} b_j\langle P_j \oplus Q_j\rangle$$

and therefore

$$P \oplus \left[Q_i^{a_i} \oplus P_i^{a_i} \oplus (P_j \oplus Q_j)^{b_j}\right] \simeq Q \oplus \left[(P_i \oplus Q_i)^{a_i} \oplus P_j^{b_j} \oplus Q_j^{b_j}\right].$$

Hence $P \oplus F \simeq Q \oplus L$ with $L = Q_i^{a_i} \oplus P_i^{a_i} \oplus P_j^{b_j} \oplus Q_j^{b_j}$ a projective R-module. Since L is projective there exist a R-module N and a free module F such that $L \oplus N \simeq F$. Therefore

$$P \oplus F \simeq P \oplus L \oplus N \simeq Q \oplus L \oplus N \simeq Q \oplus F,$$

which completes the proof. □

Definition. Let Q be a noetherian R-module. Suppose there exist free R-modules F and G such that $Q \oplus G \simeq F$, then Q is said to be stably free. Clearly projective modules are stably free.

Corollary 44. *Let R be a noetherian domain. If $K_0(R) \simeq \mathbb{Z}$, then all finitely generated projective R-modules are stably free.*

Proof. Use the rank function

$$\mathrm{rank} : K_0(R) \longrightarrow \mathbb{Z}$$

$$[R] \longmapsto 1$$

Since by assumption $K_0(R) \simeq \mathbb{Z}$ and since rank is surjective, it must be also injective and hence an isomorphism. Let Q be a finitely generated projective R-module, say $\mathrm{rank}(Q) = r$, then

$$\mathrm{rank}([Q]) = r = \mathrm{rank}([R^r]),$$

and hence $[Q] = [R^r]$ since the function rank is an isomorphism. By Theorem 43 there exists a free R-module G such that

$$Q \oplus G \simeq R^r \oplus G \simeq F \text{ a free module.}$$

Therefore Q is stably free. □

Lemma 45 (Schanuel's Lemma)**.** *Let R be a ring, and let*

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r, "i"] & P_1 \arrow[r, "\alpha"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N_2 \arrow[r, "j"] & P_2 \arrow[r, "\beta"] & M \arrow[r] & 0 \end{tikzcd}$$

be short exact sequences of R-modules, with P_1 and P_2 projective. Then $N_1 \oplus P_2 \simeq N_2 \oplus P_1$.

Proof. Since P_1 is projective there exists a map $\pi : P_2 \to P_1$ such that $\beta\pi = \alpha$. By diagram chasing we can also get a map $g : N_1 \to N_2$, so that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r, "i"] \arrow[d, "g"] & P_1 \arrow[r, "\alpha"] \arrow[d, "\pi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N_2 \arrow[r, "j"] & P_2 \arrow[r, "\beta"] & M \arrow[r] & 0 \end{tikzcd}$$

Then the following is exact:

$$0 \longrightarrow N_1 \xrightarrow{(g,i)} N_2 \oplus P_1 \xrightarrow{(j,-\pi)} P_2 \longrightarrow 0$$

If so the sequence splits as P_2 is projective, and hence the lemma follows. So we just have to prove that the above sequence is exact. Clearly (g, i) is injective, because j is injective. Also $(j, -\pi)$ is subjective: let $z \in P_2$, since α is surjective there exists $y \in P_1$ such that $\alpha(y) = \beta(z)$. But $\beta(\pi(y) - z) = \alpha(y) - \beta(z) = 0$, therefore there exists $u \in N_2$ with $j(u) = \pi(y) - z$. Finally $z = \pi(y) - j(u) = (j, \pi)(-u, -y)$. Now clearly $\mathrm{Im}(g, i) \subseteq \ker(j, -\pi)$ because $\pi \circ i = j \circ g$. Let $u \in N_2$, $y \in P_1$ such that $j(u) = \pi(y)$. Then $\alpha(y) = 0$ because $\beta(\pi) \equiv 0$, and hence $y = i(v)$ for some $v \in N_1$. But $j(g(v)) = \pi(i(v))\pi(y) = j(u)$, and since j is injective we have $u = g(v)$. Hence $(u, y) = (g, i)(v)$. □

Lemma 46 (Generalized Schanuel's Lemma)**.** *Let R be a ring and suppose we have long exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \ldots\ldots \arrow[r] & P_0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & N_2 \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \ldots\ldots \arrow[r] & Q_0 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

where $M_1 \simeq M_2$ and all P_i's and Q_j's are projective R-modules. Then

$$N_1 \oplus Q_n \oplus P_{n-1} \oplus \ldots \simeq N_2 \oplus P_n \oplus Q_{n-1} \oplus \ldots$$

Proof. Start with

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & L_0 \arrow[r] & Q_0 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

to get, by Schanuel's Lemma, that $P_0 \oplus L_0 \simeq Q_0 \oplus K_0$. Now use induction on

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \ldots\ldots \arrow[r] & P_2 \arrow[r] & P_1 \oplus Q_0 \arrow[r] & K_0 \oplus Q_0 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & N_2 \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \ldots\ldots \arrow[r] & Q_2 \arrow[r] & Q_1 \oplus P_0 \arrow[r] & L_0 \oplus P_0 \arrow[r] & 0 \end{tikzcd}$$

to get the result, again by Schanuel's Lemma. □

Theorem 47. *Let R be a regular ring,* $\dim R = d$. *Then*

$$G_0(R) \simeq K_0(R).$$

Proof. There exists an obvious map

$$\begin{aligned} i : K_0(R) &\longrightarrow G_0(R) \\ [P] &\longmapsto [P] \end{aligned}$$

Let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$, and look at its projective resolution

$$0 \longrightarrow K \longrightarrow P_{d-1} \longrightarrow P_{d-2} \longrightarrow \ldots\ldots \qquad \longrightarrow P_1 \longrightarrow P_0 M \longrightarrow 0.$$

If $\mathfrak{q} \in \mathrm{Spec}R$, then localizing we get the exact sequence

$$0 \longrightarrow K_{\mathfrak{q}} \longrightarrow (P_{d-1})_{\mathfrak{q}} \longrightarrow \ldots\ldots \longrightarrow (P_1)_{\mathfrak{q}} \longrightarrow (P_0)_{\mathfrak{q}} M_{\mathfrak{q}} \longrightarrow 0.$$

But $\mathrm{globdim}(R_{\mathfrak{q}}) \leqslant d$, therefore there exists a free $R_{\mathfrak{q}}$-resolution

$$0 \longrightarrow (F_d)_{\mathfrak{q}} \longrightarrow (F_{d-1})_{\mathfrak{q}} \longrightarrow \ldots\ldots \longrightarrow (F_1)_{\mathfrak{q}} \longrightarrow (F_0)_{\mathfrak{q}} M_{\mathfrak{q}} \longrightarrow 0.$$

By Generalized Schanuel's Lemma we get

$$K_{\mathfrak{q}} \oplus (\text{projective}) \simeq (F_d)_{\mathfrak{q}} \oplus (\text{projective}),$$

and hence $K_{\mathfrak{q}}$ is projective (and hence free) for all $\mathfrak{q} \in \mathrm{Spec}R$. Since K is a finitely generated locally free R-module it follows that it is projective. Using the above notation (set $P_d := K$) define a map

$$\begin{aligned} \widetilde{j} : H(R) &\longrightarrow K_0(R) \\ \langle M \rangle &\longmapsto [P_0] - [P_1] + \ldots + (-1)^d [P_d]. \end{aligned}$$

Notice that $\widetilde{j}$ is well defined since if we have another projective resolution

$$0 \longrightarrow Q_d \longrightarrow Q_{d-1} \longrightarrow \ldots\ldots \longrightarrow Q_1 \longrightarrow Q_0 M \longrightarrow 0,$$

then by Generalized Schanuel's Lemma we have $Q_{\text{even}} \oplus P_{\text{odd}} \simeq P_{\text{even}} \oplus Q_{\text{odd}}$, that is

$$[P_0] - [P_1] + \ldots + (-1)^d [P_d] = [Q_0] - [Q_1] + \ldots + (-1)^d [Q_d]$$

in $K_0(R)$. Now let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of R-modules and consider projective resolutions

$$P. \longrightarrow M_1 \longrightarrow 0$$

$$Q. \longrightarrow M_2 \longrightarrow 0$$

Then there exists a projective resolution $P. \oplus Q. \to M \to 0$ of M, so that

$$\widetilde{j}(\langle M \rangle) = [P_{\text{even}} \ \oplus \ Q_{\text{even}}] - [P_{\text{odd}} \ \oplus \ Q_{\text{odd}}] =$$

$$= [P_{\text{even}}] - [P_{\text{odd}}] + [Q_{\text{even}}] - [Q_{\text{odd}}] = \widetilde{j}(\langle M_1 \rangle) + \widetilde{j}(\langle M_2 \rangle).$$

Therefore $\widetilde{j}$ induces a homomorphism $j : G_0(R) \to K_0(R)$.
Claim. $i \circ j = \mathrm{id}_{G_0(R)}$ and $j \circ i = \mathrm{id}_{K_0(R)}$.

Proof of the Claim. Let P be a finitely generated projective R-module, then $i([P]) = [P] \in K_0(R)$. We have shown that the definition of j does not depend on the chosen resolution of P, so in particular we can consider

$$0 \longrightarrow P_0 = P \longrightarrow P \longrightarrow 0,$$

and hence $j([P]) = [P_0] = [P]$. Let now $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$, and consider a projective resolution

$$0 \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow P_{d-2} \longrightarrow \ldots\ldots \qquad \longrightarrow P_1 \longrightarrow P_0 M \longrightarrow 0.$$

Then $j([M]) = [P_{\text{even}}] - [P_{\text{odd}}] \in K_0(R)$, so that

$$ij([M]) = [P_{\text{even}}] - [P_{\text{odd}}] = [M] \in G_0(R).$$

□

Corollary 48. *Let k be a field and let R be the polynomial ring $k[x_1, \ldots, x_n]$. Then every finitely generated projective R-module is stably free.*

Proof. We know that R is regular, therefore by Theorem 47 we have

$$K_0(R) \simeq G_0(R) \simeq G_0(k) \simeq \mathbb{Z}.$$

Since $K_0(R) \simeq \mathbb{Z}$, by Corollary 44, we have that every finitely generated projective R-module is stably free. □

5 Construction and Properties of $K_1(R)$

Let R be a commutative ring with 1_R. Let us define two categories:

- Objects: Pairs (P, f) [respectively (F, f)], where P is a finitely generated projective R-module and $f : P \to P$ is an isomorphism [respectively F is a finitely generated free R-module and $f : F \to F$ is an isomorphism].
- Morphisms: Let us define morphisms just for the first category, for the second the definition is analogous. A morphism $h : (P, f) \to (Q, g)$ consists of an R-module homomorphism $h : P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "h"'] & P \arrow[d, "h"] \\ Q \arrow[r, "g"'] & Q \end{tikzcd}$$

In particular, if h is invertible this means that $g = hfh^{-1}$. A sequence

$$0 \longrightarrow (P_1, f_1) \xrightarrow{h} (P_2, f_2) \xrightarrow{g} (P_3, f_3) \longrightarrow 0 \qquad (\#)$$

is exact if and only if the following sequence of R-modules is exact

$$0 \longrightarrow P_1 \xrightarrow{h} P_2 \xrightarrow{g} P_3 \longrightarrow 0$$

i.e. $P_2 \simeq P_1 \oplus P_3$ since they are projective.

Definition. We define

$$K_1(R) := \frac{\text{free abelian group on } (P, f) \text{ up to isomorphism}}{H},$$

where H is the subgroup generated by the following relations:

(1) Given a short exact sequence as in (#), then we introduce a relation

$$(P_2, f_2) - (P_1, f_1) - (P_3, f_3).$$

(2) If (P, f) and (P, g) are objects, then we introduce the relation

$$(P, fg) - (P, f) - (P, g).$$

By $[P, f]$ or $[(P, f)]$ we mean the image of (P, f) in $K_1(R)$. Also, we let $K_1^f(R)$ be the same construction, but in the second category of finitely generated free modules.

Remark 38. We will prove that $K_1(R) \simeq K_1^f(R)$.

Remark 39. For any finitely generated projective module P we have $[P, 1_P] = 0$. In fact $[P, 1_P] = [P, 1_P \circ 1_P] = [P, 1_P] + [P, 1_P]$, and hence the remark follows.

Proposition 49. *Let L be a field. Then*

$$K_1(L) \simeq L^* = L \setminus \{0\}.$$

Proof. □

6 Exercises

(1) What is the kernel of α?

(2) Is the following exact?

$$G_0\left(R/(x)\right) \longrightarrow G_0\left(R\right) \longrightarrow G_0\left(R_x\right) \longrightarrow 0$$

(3) Show that sequence (2.9) on page 40 is exact.

(4) So that $G_0\left(()\,\mathbb{R}[x,y]/(x^2+y^2\right)$ is $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

Chapter 3

The Module of Differentials

Throughout this chapter, k, R, S will be commutative rings, and $k \to R \to S$ will denote ring homomorphisms, so that R and S are k-algebras and S is a R-algebra.

Definition. Let $k \to R$ be a k-algebra, and let M be a R-module. A k-derivation $D : R \to M$ is a map such that

(1) $D(r + s) = D(r) + D(s)$ for all $r, s \in R$.

(2) $D(\alpha r) = \alpha D(r)$ for all $r \in R$ and for all $\alpha \in k$.

(3) $D(rs) = rD(s) + sD(r)$ for all $r, s \in R$.

We will denote the set of all k-derivations on M by $\mathrm{Der}_k(R, M)$.

Remark 40. The definition of derivation makes sense even if the ring is not commutative, but in that case we have to respect the order of the multiplication. For instance in (3) we have to require $D(rs) = rD(s)+D(r)s$ instead of $D(rs) = rD(s) + sD(r)$.

Remark 41. $D(\alpha) = 0$ for all $\alpha \in k$.

Proof. Notice that

$$D(1) = D(1 \cdot 1) = 1 \cdot D(1) + 1 \cdot D(1),$$

and hence $D(1) = 0$. It follows that

$$D(\alpha) = \alpha D(1) = 0$$

for all $\alpha \in k$. □

Remark 42. The set $l := D^{-1}(0) = \{r \in R : D(r) = 0\}$ is a subring of R, and also D is a l-linear map. In fact l is the biggest subring $S \subseteq R$ such that D is S-linear.

Proof. If $r, s \in l$, then $D(r+s) = D(r) + D(s) = 0$. Also $D(rs) = rD(s) + sD(r) = 0$ and $D(1) = 0$, so that $l \subseteq R$ is a subring. Notice that by the previous remark we have that $k \subseteq l$. If $\beta \in l$ and $r \in R$, then

$$D(\beta r) = \beta D(r) + rD(\beta) = \beta D(r),$$

so that D is automatically a l-derivation. $\square$

Remark 43. Let D be a k-derivation and let $r \in R$. Then for $n \geqslant 1$ we have

$$D(r^n) = nr^{n-1}D(r).$$

Proof. Clearly the claim holds when $n = 1$. By induction assume it's true for $n > 1$. Then

$$D(r^{n+1}) = rD(r^n) + r^n D(r) = rnr^{n-1}D(r) + r^n D(r) = (n+1)r^n D(r).$$

$\square$

Remark 44. If $\mathrm{Char}(R) = p > 0$, then for all k-derivations we have $D(r^p) = 0$.

Remark 45. If $D : R \to M$ is a k-derivation and $f : M \to N$ is a R-homomorphism, then $f \circ D$ is a k-derivation.

We want now to construct a R-module $\Omega_{R/k}$ and a universal derivation $d : R \to \Omega_{R/k}$ such that for any other k-derivation $D : R \to M$ there exists a unique R-module homomorphism

$$\begin{tikzcd} R \arrow[rr, "d"] \arrow[dr, "D"'] & & \Omega_{R/k} \arrow[dl, dashed, "f"] \\ & M & \end{tikzcd}$$

Definition. $\Omega_{R/k}$ is called the universal module of differentials (or universal module of derivations). It is also called Kähler module of differentials.

1 First construction of the Module of Differentials

We present now a first way to contstruct $(\Omega_{R/k}, d)$. We will see a second construction, easier to deal with, later in this chapter. Take a free module on symbols $\{dr : r \in R\}$, i.e.

$$F := \bigoplus_{r \in R} Rdr.$$

We want to construct a derivation, therefore let us define $H \subseteq F$ to be the submodule generated by:

- $d(r+s) - dr - ds$.

- $d(rs) - rds - sdr$.
- $d(\alpha r) - \alpha dr$.

where $r, s \in R$, $\alpha \in k$. Then set

$$\Omega_{R/k} := F/H \qquad\qquad \begin{aligned} d : R &\longrightarrow \Omega_{R/k} \\ r &\longmapsto dr \end{aligned}$$

Notice that d is clearly a derivation. Let now $D : R \to M$ be a k-derivation, i.e. an element of $\mathrm{Der}_k(R, M)$. Consider the following diagram

$$\begin{tikzcd} & r \arrow[rr, mapsto] & & dr \\ r \arrow[ddrr, mapsto] & R \arrow[rr, "d"] \arrow[dr, "D"'] & & \Omega_{R/k} \arrow[dl, dashed, "f"] \\ & & M & \\ & & D(r) & \end{tikzcd}$$

Notice that $f(dr) = D(r)$ is forced to make the diagram commute, and also $\{dr : r \in R\}$ generates $\Omega_{R/k}$. Hence if such $f : \Omega_{R/k} \to M$ exists it has to be unique. Notice also that on F we can define freely f on the basis elements dr. Also $f(H) = 0$ because D is a derivation, therefore $f : F \to M$ induces a map $f : \Omega_{R/k} \to M$. Finally $(\Omega_{R/k}, d)$ is unique (up to isomorphism) by usual universal property arguments. If we consider $d' : R \to \Omega'$, then we have a commutative diagram

$$\begin{tikzcd} R \arrow[rr, "d'"] \arrow[dr, "D"'] & & \Omega' \arrow[dl, bend right, "g"'] \\ & \Omega_{R/k} \arrow[ur, bend right, "f"'] & \end{tikzcd}$$

and one can verify that $f \circ g = \mathrm{id}_{\Omega'}$ and $g \circ f = \mathrm{id}_{\Omega_{R/k}}$.

Remark 46. By uniqueness and by Remark 45 we have

$$\mathrm{Der}_k(R, M) \simeq \mathrm{Hom}_R(\Omega_{R/k}, M).$$

Proposition 50. *Let k be a ring and let $R = k[x_\lambda]_{\lambda \in \Lambda}$ be a polynomial ring. Then*

$$\Omega_{R/k} = \bigoplus_{\lambda \in \Lambda} R dx_\lambda$$

a free R-module. Also, for $f \in R$, we define

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_\lambda \in \Omega_{R/k},$$

where the sum involves only finitely many i's.

Proof. Let $F = \bigoplus_{\lambda \in \Lambda} R dx_\lambda$. Consider the diagram

$$\begin{tikzcd}
k[x_\lambda] \arrow[r, "d"] \arrow[dr, "D"'] & F \arrow[d, dashed] & dx_\lambda \arrow[dl, mapsto] \\
 & M & \\
 & D(x_\lambda) &
\end{tikzcd}$$

where the choice $dx_\lambda \mapsto D(x_\lambda)$ is forces by the commutativity of the diagram. F is free, hence the map is well defined. The key point of this proof is that $d : k[x_\lambda] \to F$ is in fact a derivation. For $f, g \in k[x_\lambda]$ and $\alpha \in k$ we have:

- $d(f+g) = \sum_i \frac{\partial(f+g)}{\partial x_i} dx_i = \sum_i \frac{\partial f}{\partial x_i} dx_i + \sum_i \frac{\partial g}{\partial x_i} dx_i = df + dg.$
- $d(fg) = \sum_i \frac{\partial(fg)}{\partial x_i} dx_i = f \sum_i \frac{\partial g}{\partial x_i} dx_i + g \sum_i \frac{\partial f}{\partial x_i} dx_i = f dg + g df.$
- $d(\alpha f) = \sum_i \frac{\partial(\alpha f)}{\partial x_i} dx_i = \alpha \sum_i \frac{\partial f}{\partial x_i} dx_i = \alpha df.$

□

Proposition 51. *Let R be a k-algebra and let $I \subseteq R$ be an ideal. Then*

$$\Omega_{(R/I)/k} \simeq \frac{\Omega_{R/k}}{R\langle di : i \in I\rangle},$$

with $\overline{d}(r + I) = dr + R\langle di : i \in I\rangle$.

Proof. We will prove that $(\Omega_{(R/I)/k}, \overline{d})$ satisfies the universal property. Let $D : R/I \to M$ be a k-derivation. Then

$$\begin{tikzcd}
R \arrow[rr, "d"] \arrow[dr] \arrow[ddr, bend right, "\widetilde{D}"'] & & \Omega_{R/k} \arrow[ddl, dashed, "f \in \mathrm{Hom}_R(\Omega_{R/k}, M)"] \\
 & R/I \arrow[d, "D"] & \\
 & M &
\end{tikzcd}$$

Notice that $f(di) = \widetilde{D}(i) = 0$ for all $i \in I$, therefore we get a unique induced map $\overline{d}$, which is clearly a derivation:

$$\begin{tikzcd}
R \arrow[rrr, "d"] \arrow[dr] & & & \Omega_{R/k} \arrow[dl] \\
 & R/I \arrow[r, dashed, "\overline{d}"] \arrow[d] & \frac{\Omega_{R/k}}{R\langle di : i \in I\rangle} \arrow[dl] & \\
 & M & &
\end{tikzcd}$$

□

Example 28. Let $R = k[x]/(f)$. Find $\Omega_{R/k}$. By the previous proposition we have

$$\Omega_{R/k} \simeq \frac{\Omega_{k[x]/k}}{k[x]\langle di : i \in (f)\rangle}.$$

Notice that if $i \in (f)$, then $i = fg$, and hence

$$di = fdg + gdf = fdg + gf'dx,$$

therefore $R\langle di : i \in (f)\rangle \subseteq fk[x]dx + f'k[x]dx$. Conversely

$$f'dx = df \in R\langle di : i \in (f)\rangle,$$

and also $d(xf) = xdf + fdx$, so that

$$fdx = d(xf) - xdf \in R\langle di : i \in (f)\rangle.$$

Hence $R\langle di : i \in (f)\rangle = fk[x]dx + f'k[x]dx$. This means

$$\Omega_{R/k} \simeq \frac{k[x]dx}{fk[x]dx + f'k[x]dx} \simeq \frac{k[x]}{(f, f')} \simeq \frac{R}{(f')}.$$

Definition. Let $R \to S$ be a map of algebras, and assume that S is finitely generated as an R-algebra. For every R-algebra T and for every ideal $J \subseteq T$ such that $J^2 = 0$ define the natural map

$$\theta_{T,J} : \mathrm{Hom}_R^{\mathrm{alg}}(S, T) \to \mathrm{Hom}_R^{\mathrm{alg}}(S, T/J),$$

where $\mathrm{Hom}_R^{\mathrm{alg}}(\cdot, \cdot)$ denotes the module of R-algebra homomorphisms. Then

(A) S is said to be smooth over R if $\theta_{T,J}$ is surjective for all T, J as above.

(B) S is said to be unramified over R if $\theta_{T,J}$ is injective for all T, J as above.

(C) S is said to be étale over R if it is both smooth and unramified, i.e. if $\theta_{T,J}$ is an isomorphism for all T, J as above.

We will prove the following theorem:

Theorem 52. *Let $R \to S$ be as in the above definition. Then*

(A') S is smooth over R if and only if S is flat over R and $\Omega_{S/R}$ is a projective S-module.

(B') S is unramified over R if and only if $\Omega_{S/R} = 0$.

(C') S is étale over R if and only if S is flat over R and $\Omega_{S/R} = 0$.

2 Second construction of the Module of Differentials

Let k be a ring, and let R be a k-algebra. Map

$$R \otimes_k R \xrightarrow{\mu} R$$

$$\sum_i r_i \otimes s_i \longmapsto \sum_i r_i s_i$$

and let $\mathscr{I} := \ker \mu$. We will prove that the following is an isomorphism.

$$\mathscr{I}/\mathscr{I}^2 \xrightarrow{\simeq} \Omega_{R/k}$$

$$(r \otimes 1 - 1 \otimes r) + \mathscr{I}^2 \longmapsto dr$$

Remark 47. $R \otimes_k R$ has both left and right module structures, in fact we can consider

$$r(\sum_i s_i \otimes t_i) = \sum_i (rs_i) \otimes t_i \quad \text{and} \quad (\sum_i s_i \otimes t_i)r = \sum_i s_i \otimes (rt_i).$$

Remark 48. With respect to either left or right R-module structure we have

$$\mathscr{I} = R\langle r \otimes 1 - 1 \otimes r : r \in R\rangle$$

Proof. We just prove the case with the left module structure. The other case is similar. Clearly $R\langle r \otimes 1 - 1 \otimes r : r \in R\rangle \subseteq \mathscr{I}$. Let $\sum_i r_i \otimes s_i \in \mathscr{I}$, so that $\sum_i r_i s_i = 0$. Consider

$$-\sum_i r_i(s_i \otimes 1 - 1 \otimes s_i) = \sum_i (-r_i s_i) \otimes 1 + \sum_i r_i \otimes s_i.$$

Hence $\sum_i r_i \otimes s_i \in \mathscr{I}$ and hence the remark follows. $\square$

Remark 49. The two R-module structures on $\mathscr{I}/\mathscr{I}^2$ are the same.

Proof. It is enough to show it on the generators. Let $r, s \in R$, then

$$s(r\otimes 1-1\otimes r)-(r\otimes 1-1\otimes r)s = rs\otimes 1-s\otimes r-r\otimes s+1\otimes rs = (r\otimes 1-1\otimes r)(s\otimes 1-1\otimes s) \in \mathscr{I}^2,$$

that is $s(r \otimes 1 - 1 \otimes r) = (r \otimes 1 - 1 \otimes r)s$ in $\mathscr{I}/\mathscr{I}^2$. $\square$

Remark 50. The map

$$d : R \longrightarrow \mathscr{I}/\mathscr{I}^2$$

$$r \longmapsto r \otimes 1 - 1 \otimes r$$

is a k-derivation.

Proof. Let $r, s \in R$, then

$$d(r+s) = (r+s)\otimes 1 - 1\otimes(r+s) = (r\otimes 1 - 1\otimes r) + (s\otimes 1 - 1\otimes s) = d(r) + d(s).$$

Also

$$d(rs) = (rs)\otimes 1 - 1\otimes(rs) = s(r\otimes 1 - 1\otimes r) + (s\otimes 1 - 1\otimes s)r.$$

Since in $\mathscr{I}/\mathscr{I}^2$ left and right action are the same we get $(s\otimes 1 - 1\otimes s)r = r(s\otimes 1 - 1\otimes s)$, and therefore

$$d(rs) = rd(s) + sd(r).$$

Finally, if $\alpha \in k$ and $r \in R$, we get

$$d(\alpha r) = (\alpha r)\otimes 1 - 1\otimes(\alpha r) = \alpha(r\otimes 1 - 1\otimes r) = \alpha d(r).$$

□

Before proving that $(\mathscr{I}/\mathscr{I}^2, d)$ is in fact the module of differentials we want to introduce the notion of idealization. Let R be a ring and let M be a R-module. Define

$$S := R \ltimes M := \{(r, m) : r \in R, m \in M\},$$

adding componentwise and multiplying using the following law

$$(r, m)\cdot(s, n) = (rs, rn + sm).$$

Another way to see this multiplication is "using distributive property with $M^2 = 0$":

$$(r, m)\cdot(s, n) \text{ “}=\text{” } (r+m)\cdot(s+n) = rs + rn + sm + mn = rs + rn + sm \text{ “}=\text{” } (rs, rn + sm).$$

With this choices $R \ltimes M$ is a commutative ring with identity $(1, 0)$. Another way to see this is

$$R \ltimes M = \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} : r \in R, m \in M \right\}$$

with the usual ring operations.

Remark 51. If R is a Cohen-Macaulay ring and ω_R is its canonical module, then $S = R \ltimes \omega_R$ is Gorenstein. Notice also that $S/\omega_R \simeq R$, and $\omega_R^2 = 0$ in S, hence up to radical every Cohen-Macaulay module is Gorenstein.

Question. *Let R be a domain and let M be a torsion free R-module. When does there exists an idealization $S = R \overset{\bullet}{\ltimes} M$ such that S is also a domain?*

We are now ready to prove the theorem.

Theorem 53. *With the above notation we have*

$$\begin{aligned} \mathscr{I}/\mathscr{I}^2 &\xrightarrow{\simeq} \Omega_{R/k} \\ r\otimes 1 - 1\otimes r &\longleftrightarrow dr \end{aligned}$$

Proof. We want to show that $(\mathscr{I}/\mathscr{I}^2, d)$ satisfies the universal property, i.e. if $D \in \mathrm{Der}_k(R, M)$ for some M R-module, then there exists $f : \mathscr{I}/\mathscr{I}^2 \to M$ such that

$$\begin{tikzcd} R \arrow[rr, "d"] \arrow[dr, "D"'] & & \mathscr{I}/\mathscr{I}^2 \arrow[dl, dashed, "f"] \\ & M & \end{tikzcd}$$

As usual, since $\mathscr{I}/\mathscr{I}^2 = R\langle r\otimes 1 - 1\otimes r\rangle$, if f exists it is forced to be unique, since

$$f(r\otimes 1 - 1\otimes r) = f(dr) = D(r).$$

Let $S = R \ltimes M$. There exists a k-algebra homomorphism

$$\begin{aligned} R\otimes_k R &\xrightarrow{\quad h \quad} S \\ r\otimes s &\longmapsto (rs, sD(r)) = (r, D(r))\cdot(s,0) \end{aligned}$$

and this follows from the fact that the map

$$\begin{aligned} R\otimes_k R &\xrightarrow{\quad h \quad} S \\ (r,s) &\longmapsto (rs, sD(r)) = (r, D(r))\cdot(s,0) \end{aligned}$$

is k-bilinear, and from the universal properties for tensor products. Notice that

$$h(r\otimes 1 - 1\otimes r) = h(r\otimes 1) - h(1\otimes r) = (r, D(r)) - (r,0) = (0, D(r)),$$

hence there exists a map

$$\begin{aligned} \widetilde{f} : \mathscr{I} &\longrightarrow M \\ r\otimes 1 - 1\otimes r &\longmapsto D(r) \end{aligned}$$

Also $\widetilde{f}(\mathscr{I}^2) = 0$ since $\widetilde{f}$ is a map of rings and hence

$$\widetilde{f}(\mathscr{I}^2) = \widetilde{f}(\mathscr{I})\cdot\widetilde{f}(\mathscr{I}) \subseteq M^2 = 0.$$

Therefore $\widetilde{f}$ induces the desired map $f : \mathscr{I}/\mathscr{I}^2 \to M$ that makes the above diagram commute. $\square$

3 The Jacobi-Zariski Sequence

Remark 52. Consider $k \to R \xrightarrow{\varphi} S$. Then there exists a natural S-module homomorphism

$$S \otimes_R \Omega_{R/k} \longrightarrow \Omega_{S/k}$$

Proof. There exists a R-linear map

$$\begin{aligned} \Omega_{R/k} &\longrightarrow \Omega_{S/k} \\ dr &\longmapsto d\varphi(r) \end{aligned}$$

This gives a R-bilinear map

$$S \times \Omega_{R/k} \longrightarrow \Omega_{S/k}$$

and hence we get a S-linear homomorphism

$$S \otimes_R \Omega_{R/k} \longrightarrow \Omega_{S/k}$$

□

Question. *When is the map an isomorphism? When is there a left inverse?*

Definition. We say that a R-module homomorphism $\alpha : M \to N$ is left split if there exists $\beta : N \to M$ a map such that $\beta \circ \alpha = \mathrm{id}_M$.

Remark 53. $\alpha : M \to N$ is left split if and only if for all R-modules K the induced map $\mathrm{Hom}_R(\alpha, K) : \mathrm{Hom}_R(N, K) \to \mathrm{Hom}_R(M, K)$ is onto.

Proof. Assume α is left split, so that $N \simeq M \oplus L$ via α. Then just extend any map $f : M \to K$ just by defining it to be zero on L. In other words

$$\mathrm{Hom}_R(N, K) \simeq \mathrm{Hom}_R(M, K) \oplus \mathrm{Hom}_R(L, K),$$

so that $\mathrm{Hom}_R(N, K) \to \mathrm{Hom}_R(M, K)$ is onto.

Conversely assume that the map $\mathrm{Hom}_R(\alpha, K) : \mathrm{Hom}_R(N, K) \simeq \mathrm{Hom}_R(M, K) \oplus \mathrm{Hom}_R(L, K)$ is onto for all R-modules K. Choose $M = K$, then we can lift the identity map $\mathrm{id}_M : M \to M$ to a map $\beta : N \to M$, and clearly $\mathrm{Hom}_R(\alpha, K)(\beta) = \beta \circ \alpha = \mathrm{id}_M$, so that α is left split. □

Remark 54. If M is finitely generated it is enough to check just for all K finitely generated.

Proposition 54. *Let $k \to R \xrightarrow{\varphi} S$. Then*

(1) *The natural map $S \otimes_R \Omega_{R/k} \to \Omega_{S/k}$ is left split if and only if for all S-modules N and for all $D \in \mathrm{Der}_k(R, N)$, D can be extended to a k-derivation $\widetilde{D} : S \to N$.*

(2) *The natural map $S \otimes_R \Omega_{R/k} \to \Omega_{S/k}$ is an isomorphism if and only if the extension is unique.*

Proof. (1) Rephrasing the statement we have to prove that $S \otimes_R \Omega_{R/k} \to \Omega_{S/k}$ is left split if and only if $\mathrm{Der}_k(S, N) \to \mathrm{Der}_k(R, N)$ is onto. But we know that

$$\mathrm{Der}_k(S, N) = \mathrm{Hom}_S(\Omega_{S/k}, N) \to \mathrm{Hom}_R(\Omega_{R/k}, N) = \mathrm{Der}_k(R, N).$$

By Hom-tensor adjointness we get

$$\mathrm{Hom}_S(\Omega_{S/k}, N) \to \mathrm{Hom}_R(\Omega_{R/k}, \mathrm{Hom}_S(S, N)) \simeq \mathrm{Hom}_S(\Omega_{R/k} \otimes_R S, N),$$

which is onto if and only if the map $S \otimes_R \Omega_{R/k} \to \Omega_{S/k}$ is left split by Remark 53.

(2) With the same argument used in (1) we have that the extension is unique if and only if $\mathrm{Der}_k(S, N) \simeq \mathrm{Der}_k(R, N)$, if and only if $\mathrm{Hom}_S(\Omega_{S/k}, N) \simeq \mathrm{Hom}_S(\Omega_{R/k} \otimes_R S, N)$, if and only if the map $S \otimes_R \Omega_{R/k} \to \Omega_{S/k}$ is an isomorphism, since the statement is true for all S-modules N.

□

Corollary 55. *Let $W \subseteq R$ be a multiplicatively closed set, where R is a k-algebra. Then*

$$\Omega_{R/k} \otimes_R R_W \simeq \left(\Omega_{R/k}\right)_W \simeq \Omega_{R_W/k}.$$

Proof. Consider $k \to R \to R_W$. By the above proposition the corollary holds if and only if

$$\mathrm{Der}_k(R, N) \simeq \mathrm{Der}_k(R_W, N) \quad \text{for all } N.$$

Let $D : R \to N$ be a k-derivation. Uniqueness is forced, in fact for $w \in W$:

$$D(r) = D\left(w\frac{r}{w}\right) = wD\left(\frac{r}{w}\right) + \frac{r}{w}D(w).$$

Hence

$$D\left(\frac{r}{w}\right) = \frac{D(r) - \frac{r}{w}D(w)}{w}$$

is forced. Finally note that using this equality as a definition, we get a k-derivation $D : R_W \to N$, so that $\mathrm{Der}_k(R, N) \simeq \mathrm{Der}_k(R_W, N)$ and the corollary is proved. □

Remark 55. Suppose $k \to k'$ is a ring homomorphism and suppose R is a k-algebra. Write $R' := R \otimes_k k'$. Then

$$k' \otimes_k \Omega_{R/k} \simeq \Omega_{R'/k'}.$$

Theorem 56. *Consider $k \to S \to S/I = R$ for some $I \subseteq S$ ideal. Then there exists an exact sequence*

$$\frac{I}{I^2} \xrightarrow{\overline{d}} \frac{\Omega_{S/k}}{I\Omega_{S/k}} \longrightarrow \Omega_{R/k} \longrightarrow 0$$

where $\overline{d}(i + I^2) = di + I\Omega_{S/k}$.

Proof. The last homomorphism is the natural map

$$\varphi : R \otimes_S \Omega_{S/k} \longrightarrow \Omega_{R/k},$$

but here $R = S/I$, hence

$$R \otimes_S \Omega_{S/k} \simeq \frac{\Omega_{S/k}}{I\Omega_{S/k}}.$$

Clearly the map is onto, since for all $r \in R$ there exists $s \in S$ lifting r, and hence if $dr \in \Omega_{R/k}$, then $ds \mapsto dr$. By Proposition 51 we also have

$$\Omega_{R/k} \simeq \frac{\Omega_{S/k}}{S\langle di : i \in I\rangle}.$$

Notice that $I\Omega_{S/k} \subseteq S\langle di : i \in I\rangle$, so that

$$\ker \varphi = \frac{S\langle di : i \in I\rangle}{I\Omega_{S/k}}.$$

Clearly I does surject onto $S\langle di : i \in I\rangle$ via $i \mapsto di$, but also for all $i, i' \in I$ we have $d(ii') = idi' + i'di \in I\Omega_{S/k}$, so that we get

$$\begin{tikzcd}
\frac{I}{I^2} \arrow[rr, "\overline{d}"] \arrow[dr] & & \frac{\Omega_{S/k}}{I\Omega_{S/k}} \arrow[r, "\varphi"] & \Omega_{R/k} \arrow[r] & 0 \\
& \frac{S\langle di:i\in I\rangle}{I\Omega_{S/k}} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

□

Example 29. Let k be a field of characteristic not equal to 2 or 3. Let also $S = k[x, y]$ and let $R = S/(f) = k[x, y]/(x^2 + y^3 - 1)$. We know that $\Omega_{S/k} \simeq Sdx \oplus Sdy \simeq S^2$. Also $f \in S$ is a regular sequence, so that

$$\frac{f}{f^2} \simeq S/(f) = R.$$

Also $\Omega_{S/k} \otimes_S R \simeq Rdx \oplus Rdy \simeq R^2$. Therefore we have an exact sequence

$$R \xrightarrow{(2x,3y)} R^2 \longrightarrow \Omega_{R/k} \longrightarrow 0$$

because here the map $\overline{d}$ is given by the partial derivatives of f. Since $2, 3$ are units in R we have that $(2x, 3y)R = R$, that is $(2x, 3y)$ is unimodular. Therefore we get

$$\Omega_{R/k} \oplus R \simeq R^2$$

and in particular $\Omega_{R/k}$ is projective.

Example 30. Describe $\Omega_{R/k}$ where $R = k[x,y,z]/(x^2,y^3,z^4)$, assuming 6 is a unit in k. Then we have an exact sequence

$$R^3 \xrightarrow[\begin{pmatrix} 2x & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 4z^3 \end{pmatrix}]{} R^3 \longrightarrow \Omega_{R/k} \longrightarrow 0,$$

so that

$$\Omega_{R/k} \simeq \frac{R}{(x)} \oplus \frac{R}{(y^2)} \oplus \frac{R}{(z^3)}.$$

Definition. Let $k \to S \to S/I = R$ be as above, so that we have an exact sequence

$$\frac{I}{I^2} \xrightarrow{\overline{d}} \frac{\Omega_{S/k}}{I\Omega_{S/k}} \longrightarrow \Omega_{R/k} \longrightarrow 0.$$

If $S = k[x_\lambda]$ is a polynomial ring, then we denote $\ker \overline{d} =: \Gamma_{R/k}$. One can prove that the definition does not depend on the choice of the presentation S.

Theorem 57 (Jacobi-Zariski sequence)**.** *Let $k \to R \to S$ be ring homomorphisms. Then we have an exact sequence of S-modules*

$$\Gamma_{S/k} \longrightarrow \Gamma_{S/R} \longrightarrow S \otimes_R \Omega_{R/k} \longrightarrow \Omega_{S/k} \longrightarrow \Omega_{S/R} \longrightarrow 0.$$

Furthermore, if $\Omega_{R/k}$ is flat over R, then we can add $S \otimes_R \Gamma_{R/k}$ on the left, i.e. the following sequence is exact

$$S \otimes_R \Gamma_{R/k} \longrightarrow \Gamma_{S/k} \longrightarrow \Gamma_{S/R} \longrightarrow S \otimes_R \Omega_{R/k} \longrightarrow \Omega_{S/k} \longrightarrow \Omega_{S/R} \longrightarrow 0.$$

Proof. To define $\Gamma_{\cdot/\cdot}$ we can choose any presentation for R. Let us choose

$$R = \frac{k[x_\lambda]}{I} =: \frac{A}{I} \quad \text{and} \quad S = \frac{k[x_\lambda, y_\nu]}{L} =: \frac{B}{L},$$

where $IB \subseteq L$. In this way

$$S = \frac{B}{L} = \frac{R[y_\nu]}{\overline{L}},$$

where $\overline{L} = L/IB$. Notice that there is a short exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & \frac{IB}{L^2 \cap IB} \simeq \frac{L^2+IB}{L^2} \arrow[r] & \frac{L}{L^2} \arrow[r] & \frac{\overline{L}}{\overline{L}^2} = \frac{L}{L^2+IB} \arrow[r] & 0 \\
\frac{I}{I^2} \otimes_R S \simeq \frac{IB}{LIB} \arrow[urr, two heads] & & & & &
\end{tikzcd}$$

therefore we get a commutative diagram

$$\begin{tikzcd}
 & \frac{I}{I^2}\otimes_R S \arrow[r] \arrow[d, "\overline{d}\otimes 1"] & \frac{L}{L^2} \arrow[r] \arrow[d, "\overline{d}"] & \frac{\overline{L}}{\overline{L}^2} \arrow[r] \arrow[d, "\overline{d}"] & 0 \\
0 \arrow[r] & (\bigoplus_\lambda Rdx_\lambda)\otimes_R S \arrow[r] \arrow[d] & \bigoplus_{\lambda,\nu}(Sdx_\lambda \oplus Sdy_\nu) \arrow[r] \arrow[d] & \bigoplus_\nu Sd_\nu \arrow[r] \arrow[d] & 0 \\
 & \Omega_{R/k}\otimes_R S \arrow[d] & \Omega_{R/k} \arrow[d] & \Omega_{S/R} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where

$$\frac{I}{I^2}\otimes_R S \longrightarrow \left(\bigoplus_\lambda Rdx_\lambda\right)\otimes_R S \longrightarrow \Omega_{R/k}\otimes_R S \longrightarrow 0$$

is exact because tensor product is right exact. Hence, by the Snake Lemma we get a long exact sequence

$$\ker(\overline{d}\otimes 1) \longrightarrow \Gamma_{S/k} \longrightarrow \Gamma_{S/R} \longrightarrow S\otimes_R \Omega_{R/k} \longrightarrow \Omega_{S/k} \longrightarrow \Omega_{S/R} \longrightarrow 0.$$

For the last statement assume that $\Omega_{R/k}$ is flat over R. By definition we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{R/k} \arrow[r] & \frac{I}{I^2} \arrow[rr, "\overline{d}"] \arrow[dr] & & \bigoplus_\lambda Rdx_\lambda \arrow[r] & \Omega_{R/k} \arrow[r] & 0 \\
 & & & C \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

and since $\Omega_{R/k}$ is flat and $\bigoplus_\lambda Rdx_\lambda$ is free we have that C is flat over R. Tensor the first half of the sequence to get

$$\begin{tikzcd}
\mathrm{Tor}_1^R(C,S)=0 \arrow[r] & \Gamma_{R/k}\otimes_R S \arrow[r] & \frac{I}{I^2}\otimes_R S \arrow[r] \arrow[dr, "\overline{d}\otimes 1"'] & C\otimes_R S \arrow[r] \arrow[d] & 0 \\
 & & & (\bigoplus_\lambda Rdx_\lambda)\otimes_R S &
\end{tikzcd}$$

Hence $\ker(\overline{d}\otimes 1)\simeq \Gamma_{R/k}\otimes_R S$. □

4 Quasi-unramified maps

Definition. Let $k\to R$ be a ring homomorphism. R is quasi-unramified over k if for all k-algebras T and for all ideals $J\subseteq T$ such that $J^2=0$ the map

$$\mathrm{Hom}_k^{\mathrm{alg}}(R,T)\to \mathrm{Hom}_k^{\mathrm{alg}}(R,T/J)$$

is injective.

Remark 56. The definition is the same as the one of unramified, except for the fact that we are not assuming that R is a finitely generated k-algebra.

Theorem 58. *Let $k \to R$ be a ring homomorphism. Then R is quasi-unramified if and only if $\Omega_{R/k} = 0$.*

Proof. Recall that $\Omega_{R/k} \simeq \mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is given by

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I} \arrow[r] & R \otimes_k R \arrow[r] & R \arrow[r] & 0 \\
 & & r \otimes s \arrow[r] & rs &
\end{tikzcd}$$

Suppose R is quasi-unramified. Set $T = R \otimes_k R/\mathscr{I}^2$ and $J = \mathscr{I}/\mathscr{I}^2 \subseteq T$, so that $T/J \simeq R \otimes_k R/\mathscr{I} \simeq R$. Consider the following diagram

$$\begin{tikzcd}
R \arrow[rr, bend left, "r \mapsto r \otimes 1"] \arrow[rr, bend right, "r \mapsto 1 \otimes r"'] \arrow[dr, "\mathrm{id}"'] & & T \arrow[dl] \\
 & T/J \arrow[dl] & \\
0 & &
\end{tikzcd}$$

Then we have to liftings of the identity, hence they have to coincide. Hence $r \otimes 1 + \mathscr{I}^2 = 1 \otimes r + \mathscr{I}^2$, i.e.

$$r \otimes 1 - 1 \otimes r \in \mathscr{I}^2.$$

These elements generate $\mathscr{I}$, therefore $\mathscr{I} = \mathscr{I}^2$ and

$$\Omega_{R/k} \simeq \frac{\mathscr{I}}{\mathscr{I}^2} = 0.$$

Conversely assume $\Omega_{R/k} = 0$ and suppose we have a diagram

$$\begin{tikzcd}
R \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\beta"'] \arrow[dr] & & T \arrow[dl] \\
 & T/J &
\end{tikzcd}$$

We want to prove that $\alpha = \beta$. Consider the ring homomorphism

$$\begin{tikzcd}
R \otimes_k R \arrow[r, "\psi"] & T \\
r \otimes s \arrow[r, mapsto] & \alpha(r)\beta(s)
\end{tikzcd}$$

Notice that $\psi(r\otimes 1)-\psi(1\otimes r)=\alpha(r)-\beta(r)\in J$ because $\alpha=\beta \bmod J$. Hence $\psi(\mathscr{I})\subseteq J$ and therefore $\psi(\mathscr{I}^2)=\psi(\mathscr{I})\cdot\psi(\mathscr{I})\subseteq J^2=0$. But $\mathscr{I}/\mathscr{I}^2\simeq \Omega_{R/k}=0$, and hence $\psi(\mathscr{I})=\psi(\mathscr{I}^2)=0$. Therefore, for all $r\in R$

$$\alpha(r)-\beta(r)=\psi(r\otimes 1-1\otimes r)=0,$$

that is $\alpha=\beta$. □

Corollary 59. *Let $k\to R\to S$ be ring homomorphisms. Then*

(1) If S/k (i.e. S is unramified over k) is unramified, then so is S/R.

(2) (Transitivity) If R/k and S/R are unramified, then so is S/k.

Proof. Use Jacobi-Zariski sequence and the previous theorem. □

Example 31. Let k be a field and let $k\subseteq \ell$ be a finite separable field extension. Then there exists a primitive element, say

$$\ell=k(\alpha)\simeq\frac{k[x]}{(f(x))},$$

with $f'(\alpha)\neq 0$. We have an exact sequence

$$\begin{tikzcd}
\frac{f(x)}{f^2(x)} \arrow[r] \arrow[d, "\simeq"'] & \ell dx \arrow[r] \arrow[d, "\simeq"'] & \Omega_{\ell/k} \arrow[r] \arrow[d, equal] & 0 \\
\ell \arrow[r, "f'(\alpha)"'] & \ell \arrow[r] & \Omega_{\ell/k} \arrow[r] & 0
\end{tikzcd}$$

Then $\Omega_{\ell/k}=0$ and therefore ℓ/k is unramified.

Example 32. Let $W\subseteq R$ be a multiplicatively closed set. We have already seen that $\Omega_{R_W/R}=0$, hence R_W/R is unramified.

Example 33. Let $k\to k[x]$. We know that

$$\Omega_{k[x]/k}=k[x]dx\simeq k[x],$$

therefore $k[x]$ is not unramified over k.

Example 34. Let R be a ring and let $I\subseteq R$ be an ideal. One can easily prove that $\Omega_{(R/I)/R}=0$, hence the map $R\to R/I$ is unramified.

Example 35 (Base Change)**.** Let $k\to R$ be quasi-unramified, and let $k\to k'$ be another ring homomorphism. Then

$$\Omega_{R/k}\otimes_k k'\simeq\Omega_{R\otimes_k k'/k'},$$

therefore $k'\to R$ is quasi-unramified too.

Definition (Separable algebraic)**.** Let k be a field, and let $\overline{k}$ be its algebraic closure. Then $\alpha \in \overline{k}$ is said to be separable over k if $(f, f') = 1$, where $f(x) = \min(\alpha, k)$ the minimal polynomial of α over k. A filed extension $k \subseteq l \subseteq \overline{k}$ is separable over k if every $\alpha \in l$ is separable over k.

Remark 57. $(f, f') = 1$ if and only if $f(x)$ has distinct roots in $\overline{k}[x]$.

Remark 58. Let $k \subseteq k' \subseteq l \subseteq \overline{k}$ be field extensions, and assume l is separable over k. Then l is separable over k', because the minimal polynomial of α over k' divides the minimal polynomial of α over k.

Remark 59. Let $k \subseteq l = k(\alpha)$ be a simple algebraic field extension. Then l is separable over k if and only if $k \to l$ is unramified.

Theorem 60. *Let $k \subseteq \ell \subseteq \overline{k}$ be field extensions, with $k \subseteq \ell$ finite. Then the following facts are equivalent:*

(1) ℓ is separable over k.

(2) $\Omega_{\ell/k} = 0$.

(3) $\Gamma_{\ell/k} = 0$.

Proof. (2) $\iff$ (3) We can write

$$\ell = \frac{k[x_1, \dots, x_n]}{\mathfrak{m}},$$

where $\mathfrak{m}$ is a maximal ideal, and we know that

$$\dim_\ell \frac{\mathfrak{m}}{\mathfrak{m}^2} = n,$$

since $k[x_1, \dots, x_n]$ is regular. There is an exact sequence of ℓ-vector spaces

$$0 \longrightarrow \Gamma_{\ell/k} \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \xrightarrow{\bar{d}} \bigoplus_{i=1}^{n} \ell dx_i \longrightarrow \Omega_{\ell/k} \longrightarrow 0.$$

Looking at the dimensions we get

$$\dim_\ell \Omega_{l/k} = \dim_\ell \Gamma_{\ell/k},$$

and this implies the equivalence between (2) and (3).
(2) + (3) $\Rightarrow$ (1) Let $\alpha \in \ell$, and consider $k \subseteq k(\alpha) \subseteq \ell$. Using the Jacobi-Zariski sequence we get

$$\Gamma_{\ell/k(\alpha)} \longrightarrow \Omega_{k(\alpha)/k} \otimes_{k(\alpha)} \ell \qquad \Omega_{\ell/k} \longrightarrow \Omega_{\ell/k(\alpha)} \longrightarrow 0.$$

By assumption $\Omega_{\ell/k} = 0$, and hence $\Omega_{\ell/k(\alpha)} = 0$. But since we have already shown that (2) $\Rightarrow$ (3) we have also that $\Gamma_{\ell/k(\alpha)} = 0$, and therefore

$$\Omega_{k(\alpha)/k} \otimes_{k(\alpha)} \ell = 0.$$

Since $k(\alpha) \to \ell$ is faithfully flat (it is just a field extension) we have $\Omega_{k(\alpha)/k} = 0$, and hence α is separable over k.
(1) $\Rightarrow$ (2) Use induction on $m = [\ell : k]$. The case $m = 1$ is clear. Now assume $m > 1$, and choose $\alpha \in \ell \setminus k$. Then ℓ is separable over $k(\alpha)$ by transitivity. By induction $\Omega_{\ell/k(\alpha)} = 0$, and also $\Omega_{k(\alpha)/k} = 0$ since α is separable over k. Then, using the Jacobi-Zariski sequence we get

$$\Omega_{k(\alpha)/k} \otimes_{k(\alpha)} \ell = 0 \longrightarrow \Omega_{\ell/k} \longrightarrow \Omega_{\ell/k(\alpha)} = 0,$$

and therefore $\Omega_{\ell/k} = 0$. □

Theorem 61. *Let k be a field and let R be a finitely generated k-algebra. Then the following are equivalent:*

(1) $k \to R$ is unramified.

(2) $R \simeq k_1 \times \ldots \times k_r$, where each k_i is field which is finite and separable over k.

Proof. (2) $\Rightarrow$ (1) One can prove that if $R \simeq k_1 \times \ldots \times k_r$, then

$$\Omega_{R/k} \simeq \bigoplus_{i=1}^{r} \Omega_{k_i/k}.$$

Since each k_i is finite and separable over k, by Theorem 60 we have $\Omega_{k_i/k} = 0$ for all $i = 1, \ldots, r$, and hence $\Omega_{R/k} = 0$.
(1) $\Rightarrow$ (2) Let $\mathfrak{m} \in \operatorname{MaxSpec}(R)$. Then $R \to R/\mathfrak{m}^2$ is unramified, and by transitivity $k \to \mathfrak{m}/\mathfrak{m}^2$ is also unramified. Call $A = R/\mathfrak{m}^2$ and call $\mathfrak{m}_A = \mathfrak{m}/\mathfrak{m}^2$ its maximal ideal. Since R is a finitely generated k-algebra, by Nullstellensatz $l = A/\mathfrak{m}_A$ is a finite algebraic extension of k. In particular $\dim_k A < \infty$, because $[l : k] < \infty$ and A is artinian, hence $\dim_l A < \infty$. Consider

$$0 \longrightarrow \mathscr{I} \longrightarrow A \otimes_k A \longrightarrow A \longrightarrow 0,$$

so that

$$\mathscr{I}/\mathscr{I}^2 \simeq \Omega_{A/k} = 0.$$

Then $\mathscr{I} = \mathscr{I}^2$ and it is finitely generated, hence $\mathscr{I} = (e)$, where $e^2 = e$ is an idempotent. Assume now $k = \overline{k}$ is algebraically closed, so that $l = k$. Notice that

$$\frac{A \otimes_k A}{\mathfrak{m}_A \otimes_k A + A \otimes_k \mathfrak{m}_A} \simeq (A \otimes_k A) \otimes_A l = (A \otimes_k A) \otimes_A k = k \otimes_k k = k,$$

is a field, therefore $\mathfrak{m}_A \otimes_k A + A \otimes_k \mathfrak{m}_A$ is a maximal ideal in $A \otimes_k A$, but it is also nilpotent since $\mathfrak{m}_A^2 = 0$. This means that $A \otimes_k A$ has only one maximal ideal, and therefore it is local. But a local ring has no trivial idempotents, and hence $e = 0, 1$. Clearly $e \neq 1$, otherwise $A = 0$. So $\mathscr{I} = 0$ and therefore $A \otimes_k A \simeq A$. But then

$$\dim_k A = (\dim_k A)^2,$$

which implies $\dim_k A = 1$, and then $\mathfrak{m}_A = 0$, which is $\mathfrak{m} = \mathfrak{m}^2$. This again implies that $\mathfrak{m}$ is generated by an idempotent. Since R is noetherian, and $\mathfrak{m}$ was an arbitrary maximal ideal in R, R must have only finitely many maximal ideals (a noetherian ring cannot have infinitely many idempotents) and also

$$R \simeq k \times \ldots \times k,$$

proving the theorem in the case $k = \overline{k}$. Back to the general case, by base change we have

$$\Omega_{(R \otimes_k \overline{k})/\overline{k}} \simeq \Omega_{R/k} \otimes_k \overline{k} = 0,$$

so that $R \otimes_k \overline{k} \simeq \overline{k} \times \ldots \times \overline{k}$ (r copies) by what we have already proved. Therefore

$$\dim_k R = \dim_{\overline{k}} R \otimes_k \overline{k} = r.$$

Moreover $k \hookrightarrow \overline{k}$ is of course a flat extension, therefore

$$R = R \otimes_k k \hookrightarrow R \otimes_k \overline{k}.$$

This implies that R is reduced, and hence is a direct product of fields:

$$R \simeq k_1 \times \ldots \times k_s,$$

where each $k_i \subseteq \overline{k}$ is finite over k and $\sum_{i=1}^{s} [k_i : k] = r$. Finally, since $0 = \Omega_{R/k} = \bigoplus_{i=1}^{s} \Omega_{k_i/k}$ we have that each $\Omega_{k_i/k} = 0$ for all $i = 1, \ldots, s$ and hence each k_i is separable over k by Theorem 60. □

Proposition 62. *Let $\varphi : R \to S$ be a homomorphism of noetherian rings, and assume S is a finitely generated R-algebra. Then the following facts are equivalent:*

(1) $R \to S$ is unramified.

(2) For all $Q \in \operatorname{Spec}(S)$, set $\mathfrak{q} = \varphi^{-1}(Q)$, then $k(\mathfrak{q}) \to S \otimes_R k(\mathfrak{q})$ is unramified.

Proof. (1) ⇒ (2) Let $Q \in \operatorname{Spec}(S)$ and let $\mathfrak{q}$ be as above. By base change we get

$$\Omega_{(S \otimes_R k(\mathfrak{q}))/k(\mathfrak{q})} \simeq \Omega_{S/R} \otimes_R k(\mathfrak{q}) = 0$$

since $R \to S$ is unramified. Therefore $S \otimes_R k(\mathfrak{q})$ is unramified for all $Q \in \operatorname{Spec}(S)$.

(2) ⇒ (1) Since S is a finitely generated R-algebra we have that $\Omega_{S/R}$ is a finitely generated S-module. Then to show $\Omega_{S/R} = 0$ it is enough to show that $(\Omega_{S/R})_Q = 0$ for all $Q \in \operatorname{Spec}(S)$, and hence it is enough to show $(\Omega_{S/R})_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \operatorname{Spec}(R)$ since

$$(\Omega_{S/R})_{\mathfrak{q}} = (\Omega_{S/R})_{\varphi(R \smallsetminus \mathfrak{q})}$$

and then

$$(\Omega_{S/R})_Q = \left((\Omega_{S/R})_{\varphi(R \smallsetminus \mathfrak{q})}\right)_Q$$

is just a further localization. By base change

$$(\Omega_{S/R})_{\mathfrak{q}} = \Omega_{S/R} \otimes_R R_{\mathfrak{q}} \simeq \Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}.$$

$\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}$ is finitely generated, and hence by NAK it is enough to show that

$$\frac{\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}}{\mathfrak{q}\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}} = 0.$$

Finally, by base change

$$\frac{\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}}{\mathfrak{q}\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}} \simeq \left(\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{q}}}\right) \otimes_{R_{\mathfrak{q}}} \frac{R_{\mathfrak{q}}}{\mathfrak{q}R_{\mathfrak{q}}} \simeq \Omega_{(S\otimes_R k(\mathfrak{q}))/k(\mathfrak{q})} = 0.$$

□

Theorem 63. *Let $R \to S$ be a homomorphism of noetherian rings, and assume S is a finitely generated R-algebra. Then the following facts are equivalent:*

(1) $R \to S$ is unramified.

(2) For all $\mathfrak{q} \in \operatorname{Spec}(R)$, $S \otimes_R k(\mathfrak{q}) \simeq k_1 \times \ldots \times k_{r_{\mathfrak{q}}}$, where each k_1 is finite and separable over $k(\mathfrak{q})$.

Proof. Follows immediately from Theorem 61 and Proposition 62. □

Theorem 64 (Local Structure Theorem)**.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be essentially of finite type (i.e. S is the localization of a finitely generated R-algebra). Assume $R \to S$ is quasi-unramified. Then there exists a R-algebra*

$$T = \left(\frac{R[x]}{(f(x))}\right)_Q$$

such that

(1) f is monic and $f'(x) \in Q$.

(2) There exists a surjective map $R \to T \twoheadrightarrow S$ such that

$$\frac{T}{\mathfrak{m}_R T} \simeq \frac{S}{\mathfrak{m}_S}.$$

Example 36. Consider $\mathbb{C} \to \mathbb{C}[x]$ and let $\mathfrak{q} = (0)$. Then

$$\mathbb{C}[x] \otimes_{\mathbb{C}} k(\mathfrak{q}) = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[x],$$

which is not finite over $\mathbb{C}$. Then $\mathbb{C} \to \mathbb{C}[x]$ is not unramified.

Example 37. Consider $\mathbb{C}[x] \to \frac{\mathbb{C}[x,y]}{(x^2-y^3)} \simeq \mathbb{C}[t^2,t^3]$. Prime ideals in $\mathbb{C}[x]$ are (0) and $(x-\alpha)$, with $\alpha \in \mathbb{C}$. First let us check $\mathfrak{q} = (0)$:

$$\mathbb{C}(X) \to \left(\frac{\mathbb{C}[x,y]}{(x^2-y^3)}\right)_{\mathfrak{q}} = \frac{\mathbb{C}(x)[y]}{(x^2-y^3)}.$$

The polynomial $x^2 - y^3$ is irreducible in $\mathbb{C}(x)[y]$, hence $\mathbb{C}(X) \to \frac{\mathbb{C}(x)[y]}{(x^2-y^3)}$ is unramified. Let now $\mathfrak{q} = (x-\alpha)$, so that we get

$$\mathbb{C} = \mathbb{C} \otimes k((x-\alpha)) \longrightarrow \left(\frac{\mathbb{C}[x,y]}{(x^2-y^3)}\right) \otimes k((x-\alpha)) \simeq \frac{\mathbb{C}[y]}{(\alpha^2-y^3)}.$$

If $\alpha = 0$, then this extension is not unramified, so that $\mathbb{C}[x] \to \frac{\mathbb{C}[x,y]}{(x^2-y^3)}$ itself is not unramified. Notice that to make it unramified it is enough to invert y or y^2, i.e. to consider

$$\mathbb{C}[x] \to \left(\frac{\mathbb{C}[x,y]}{(x^2-y^3)}\right)_y$$

or similarly

$$\mathbb{C}[x] \to \left(\frac{\mathbb{C}[x,y]}{(x^2-y^3)}\right)_{y^2}.$$

5 Quasi-smooth maps

Definition. Let $k \to R$ be a ring homomorphism. R is quasi-smooth over k if for all k-algebras T and for all ideals $J \subseteq T$ such that $J^2 = 0$ the map

$$\operatorname{Hom}_k^{\text{alg}}(R,T) \to \operatorname{Hom}_k^{\text{alg}}(R,T/J)$$

is surjective. R is quasi-étale over k if $k \to R$ is quasi-unramified and quasi-smooth.

Remark 60. As for quasi-unramified and unramified, these definitions the same as the ones of smooth and étale, except for the fact that we are not assuming that R is a finitely generated k-algebra.

Theorem 65. *Let $k \to R$ be ring homomorphism. Then it is quasi-smooth if and only if*

$$\Gamma_{R/k} = 0 \quad \textit{and} \quad \Omega_{R/k} \textit{ is a projective } R-\textit{module.} \tag{3.1}$$

Proof. We want to find a condition equivalent to (3.1). Write $R = k[x_\lambda]/I$, then we have to following defining sequence:

$$0 \longrightarrow \Gamma_{R/k} \longrightarrow \frac{I}{I^2} \xrightarrow{\overline{d}} \bigoplus Rdx_i \longrightarrow \Omega_{R/k} \longrightarrow 0, \tag{3.2}$$

Hence (3.1) holds if and only if there exists $\psi : \bigoplus Rdx_i \to I/I^2$ splitting map for $\overline{d}$.

Claim. (3.2) holds if and only if

$$\exists \delta_\lambda \in I \text{ such that } f(x_\lambda - \delta_\lambda) \in I^2 \forall f \in I. \tag{3.3}$$

Proof of the Claim. Assume (3.2) holds. This is true if and only if there exists $\psi : \bigoplus Rdx_i \to I/I^2$ such that for $f \in I$ we get

$$\psi\left(\sum)\frac{\partial f}{\partial x_\lambda}dx_\lambda\right) = f + I^2,$$

that is if and only if there exists $\delta_\lambda = \psi(dx_\lambda)$ such that

$$\sum \frac{\partial f}{\partial x_\lambda}\delta_\lambda = f(x_\lambda) + I^2.$$

Recall now that

$$f(x_1,\ldots,x_n) = f(\alpha_1,\ldots,\alpha_n) + \sum \frac{\partial f}{\partial x_j}(x_j - \alpha_j) \mod (x_1-\alpha_1,\ldots,x_n-\alpha_n)^2.$$

Set $\alpha_\lambda = x_\lambda - \delta_\lambda$, so that

$$f(x_1,\ldots,x_n) \equiv f(x_\lambda - \delta_\lambda) + \sum \frac{\partial f}{\partial x_\lambda}\delta_\lambda \mod (\delta_\lambda)^2.$$

Since $\delta_\lambda = \psi(dx_\lambda) \in I$ we have that $(\delta_\lambda)^2 \subseteq I^2$. But also $f(x_\lambda) = \sum \frac{\partial f}{\partial x_\lambda}\delta_\lambda$ mod I^2. Hence we get

$$f(x_\lambda - \delta_\lambda) \in I^2.$$

Conversely assume $f(x_\lambda - \delta_\lambda) \in I^2$, then we can define $\psi : dx_\lambda \mapsto \delta_\lambda + I^2$.

So it is enough to prove the equivalence quasi-smooth if and only if (3.3). Assume (3.3) holds, then consider a diagram

$$\begin{tikzcd} k[x_\lambda] \arrow[d, two heads, "\pi"'] & T \arrow[d, two heads] \\ R \arrow[r, "\varphi"'] & \frac{T}{J} \end{tikzcd}$$

with $J^2 = 0$. Let $r_\lambda = \pi(x_\lambda) = x_\lambda + I$, and $\varphi(r_\lambda) = t_\lambda + J$. $k[x_\lambda]$ is just a polynomial ring, therefore we can define a map

$$\begin{tikzcd} k[x_\lambda] \arrow[r, "\phi"] & T \\ x_\lambda \arrow[r, mapsto] & t_\lambda \end{tikzcd}$$

We need to find $\varepsilon_\lambda \in J$ such that for all $f \in I$, $f(t_\lambda - \varepsilon_\lambda) = 0$. If we can find such ε_λ then we can send $r_\lambda \mapsto t_\lambda - \varepsilon_\lambda$ and get the required lifting of φ. Set $\varepsilon_\lambda = \phi(\delta_\lambda) \in J$, because $\phi(I) \subseteq J$ by commutativity of the following diagram

$$\begin{tikzcd} k[x_\lambda] \arrow[r, "\phi"] \arrow[d, two heads, "\pi"'] & T \arrow[d, two heads] \\ R \arrow[r, "\varphi"'] & \frac{T}{J} \end{tikzcd}$$

Finally, since by assumption we have $f(x_\lambda - \delta_\lambda) \in I^2$, we get

$$f(t_\lambda - \varepsilon_\lambda) = \phi(f(x_\lambda - \delta_\lambda)) \in J^2 = 0.$$

Conversely assume $k \to R$ is quasi-smooth, then construct δ_λ as follows. If $R = k[x_\lambda]/I$ set $T = k[x_\lambda]/I^2$, and if $J = I/I^2$ then $T/J \simeq R$. By assumption we get a diagram

$$\begin{tikzcd} & T \arrow[d, two heads] \\ R \arrow[ur, dashed, "\varphi"] \arrow[r, "\mathrm{id}_R"'] & \frac{T}{J} = R \end{tikzcd}$$

Therefore $\varphi(x_\lambda + I) = x_\lambda - \delta_\lambda$ for some $\delta_\lambda \in I$. This implies, for $f \in I$:

$$0 = \varphi(f) = f(x_\lambda - \delta_\lambda) \in T,$$

so that $f(x_\lambda - \delta_\lambda) \in I^2$. $\square$

Exercise 1. Let $W \subseteq S$ be a multiplicatively closed set, $R \to S$ a ring homomorphism. Then

$$\left(\Gamma_{S/R}\right)_W \simeq \Gamma_{S_W/R}$$

Corollary 66 (of Theorem 65)**.** *A ring homomorphism $R \to S$ is quasi-étale if and only if $\Gamma_{S/R} = \Omega_{S/R} = 0$.*

Proof. Quasi-étale if and only if quasi-smooth and quasi-unramified, if and only if

$$\begin{cases} \Gamma_{S/R} = 0 \text{ and } \Omega_{S/R} \text{ is projective (quasi-smooth)} \\ \\ \Omega_{S/R} = 0 \text{ (quasi-unramified)} \end{cases}$$

which is clearly equivalent to the condition $\Gamma_{S/R} = \Omega_{S/R} = 0$. $\square$

Corollary 67 (Transitivity)**.** *Let $k \to R \to S$ be ring homomorphisms. If $k \to R$ and $R \to S$ are (quasi-)smooth, then so is $k \to S$.*

Proof. Use the Jacobi-Zariski sequence:

$$\Gamma_{R/k} \otimes_R S \longrightarrow \Gamma_{S/k} \longrightarrow \Gamma_{S/R} \longrightarrow \Omega_{R/k} \otimes_R S \longrightarrow \Omega_{S/k} \longrightarrow \Omega_{S/R} \longrightarrow 0,$$

where the first map comes from the fact that $k \to R$ is quasi-smooth, and hence $\Omega_{R/k}$ is projective, and in particular flat. Also, since $R \to S$ is quasi-smooth, we have $\Gamma_{S/R} = 0$ and $\Omega_{S/R}$ is projective. Therefore the final part of sequence above splits:

$$\Omega_{S/k} \simeq \Omega_{S/R} \bigoplus \left(\Omega_{R/k} \otimes_R S\right).$$

Moreover, $\Omega_{R/k}$ is a projective R-module, and as a consequence $\Omega_{R/k} \otimes_R S$ is a projective S-module. This shows that $\Omega_{S/k}$ is a projective S-module. Finally, $\Gamma_{S/R} = \Gamma_{R/k} = 0$ since $k \to R$ and $R \to S$ are quasi-smooth, therefore $\Gamma_{S/k} = 0$ by the sequence above. By Theorem 65 we get that $k \to S$ is quasi-smooth. $\square$

Theorem 68 (Cartier-Mac Lane)**.** *Let k be a field and let $k \subseteq l$ be a finitely generated field extension. Then*

$$\dim_\ell \Omega_{l/k} = \dim_\ell \Gamma_{l/k} + \mathrm{tr.deg}_k \ell.$$

Proof. We have discussed the case in which $\mathrm{tr.deg}_k \ell < \infty$, i.e. when ℓ is algebraic over k. Now assume $n = \mathrm{tr.deg}_k \ell > 0$, and fix a transcendence basis $x_i, \ldots, x_n$ of ℓ over k. Set $E := k(x_1, \ldots, x_n)$, and consider the inclusions $k \subseteq E \subseteq \ell$, where $E \subseteq \ell$ is now algebraic. Then we have

$$\Gamma_{E/k} \otimes_E \ell \longrightarrow \Gamma_{\ell/k} \longrightarrow \Gamma_{\ell/E} \longrightarrow \Omega_{E/k} \otimes_E \ell \longrightarrow \Omega_{\ell/k} \longrightarrow \Omega_{\ell/E} \longrightarrow 0,$$

where the first map comes from the fact that $\Omega_{E/k}$ is clearly flat, being E a field. Consider $k \to E$, which is a purely transcendental extension. Notice that, if we set $R := k[x_1, \ldots, x_n]$ and $W = R \setminus \{0\}$, then we have that $E = R_W$. Since R is a polynomial ring over k we have that

$$\Omega_{R/k} \simeq \bigoplus_{i=1}^{n} R\mathrm{d}x_i,$$

and in particulare it is a free R-module. Localizing at W we get that

$$\Omega_{E/k} \simeq \bigoplus_{i=1}^{n} E dx_i.$$

Also, $\Gamma_{R/k} = 0$ since we can just choose $I = 0$ in the presentation of R as an algebra $k[x_1, \ldots, x_n]/I$ over k. Putting things together, and going back to the Jacobi-Zariski sequence we get

$$0 \longrightarrow \Gamma_{\ell/k} \longrightarrow \Gamma_{\ell/E} \longrightarrow \ell^n \longrightarrow \Omega_{\ell/k} \longrightarrow \Omega_{\ell/E} \longrightarrow 0,$$

because

$$\Omega_{E/k} \otimes_E \ell \simeq \left(\bigoplus_{i=1}^{n} E\mathrm{d}x_i\right) \otimes_E \ell \simeq \bigoplus_{i=1}^{n} \ell\mathrm{d}x_i \simeq \ell^n.$$

Every module is finitely generated in the sequence, therefore dimension over l are finite. From the sequence we get

$$\dim_\ell \Omega_{\ell/E} + n + \dim_\ell \Gamma_{l/k} = \dim_\ell \Omega_{\ell/k} + \dim_\ell \Gamma_{\ell/E}.$$

From the algebraic case, since $E \subseteq \ell$ is algebraic, we know that $\dim_\ell \Omega_{\ell/E} = \dim_\ell \Gamma_{\ell/E}$, therefore

$$\mathrm{tr.deg}_k \ell + \dim_\ell \Gamma_{\ell/k} = n + \dim_\ell \Gamma_{\ell/k} = \dim_\ell \Omega_{\ell/k}.$$

□

Chapter 4

Basic element theory

1 Basic sets and basic elements

Definition. Let R be a commutative ring with 1_R. A subset $X \subseteq \mathrm{Spec}R$ is said to be basic if

(i) X is Noetherian (i.e. it has DCC on closed sets)

(ii) If $\mathfrak{p}_\alpha \in X$ for $\alpha \in \Lambda$, and $\bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \in \mathrm{Spec}R$, then $\bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \in X$.

Remark 61. Finite intersection of primes which are not nested are never prime. Therefore the interesting cases in (ii) of the above definition happen when the intersection is infinite.

Remark 62. If R is Noetherian, then $\mathrm{Spec}R$ is Noetherian. But the converse does not hold in general.

Example 38. (1) When R is Noetherian, $X = \mathrm{Spec}R$ clearly is basic.

(2) When R is Noetherian,

$$X = j - \mathrm{Spec}R = \left\{ \mathfrak{p} \in \mathrm{Spec}R \mid \mathfrak{p} = \bigcap_{\substack{\mathfrak{p} \subseteq \mathfrak{m} \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m} \right\}$$

is basic.

(3) When R is Noetherian,

$$X^i := \{\mathfrak{p} \in \mathrm{Spec}R \mid \mathrm{ht}\mathfrak{p} \leqslant i\}$$

is basic.

(4) If X is basic and $F \subseteq \mathrm{Spec}R$ is closed, then $X \cap F$ is basic.

Discussion. Closed sets in the Zariski topology on R are of the form

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}R \mid I \subseteq \mathfrak{p}\} = V(\sqrt{I}) \quad \longleftrightarrow \quad \mathrm{Spec}R/I.$$

It's the weakest topology that makes ring homomorphism continuous, that is if $\varphi : R \to S$ is a ring homomorphism, then there is an induced map $\varphi^* : \mathrm{Spec}S \to \mathrm{Spec}R$ on spectra, and it has continuous in the Zariski topology. A basis of open sets is given by $\{D(f)\}_{f\in R}$, where

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}R \mid f \notin \mathfrak{p}\} \quad \longleftrightarrow \quad \mathrm{Spec}R_f.$$

Definition. Let X be basic, and let $\mathfrak{p} \in X$. Define

$$\dim_X \mathfrak{p} := \sup\{n \mid \exists \mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n, \text{ with } \mathfrak{p}_i \in X \forall i\}.$$

If $X = \mathrm{Spec}R$ then notice that $\dim_X \mathfrak{p} = \dim(R/\mathfrak{p})$, the Krull dimension.

Proposition 69. *Let X be basic. Then*

(i) Every closed set $F \subseteq X$ is a finite union of irreducible closed sets in X. Recall that a set is irreducible if it cannot be written as a proper union of smaller subsets.

(ii) If $F \subseteq X$ is not empty, closed and irreducible, then F has a generic point in X, i.e. $F = V(\mathfrak{p}) \cap X$ for some $\mathfrak{p} \in X$.

Proof. (i) It follows from the Noetherian property: given $F \subseteq X$ closed, if it is irreducible we are done. If it is reducible we can write it as a union of two smaller closed sets. Then repeat of the smaller sets, and the process ends by DCC.

(ii) Write $F = V(I) \cap X$ for some ideal $I \subseteq R$. Set

$$\mathfrak{p}_0 = \bigcap_{\mathfrak{p}\in F} \mathfrak{p}.$$

Notice that $\mathfrak{p}_0 \in X$ since X is basic, and also $F = V(\mathfrak{p}_0) \cap X$. In fact $F \subseteq V(\mathfrak{p}_0) \cap X$, because if $\mathfrak{p} \in F$, then $\mathfrak{p} \supseteq \mathfrak{p}_0$. On the other hand $V(p_0) \cap X \subseteq V(I) \cap X = F$ since $I \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in V(\mathfrak{p}_0)$ by definition of $\mathfrak{p}_0$. Hence, without loss of generality we may assume $I = \mathfrak{p}_0$.We want to prove that $\mathfrak{p}_0$ is prime. If not there exist $ab \in \mathfrak{p}_0$ such that $a \notin \mathfrak{p}_0$ and $b \notin \mathfrak{p}_0$. Set $F_1 := V(\mathfrak{p}_0, a) \cap X$ and $F_2 := V(\mathfrak{p}_0, b) \cap X$. Then $F = F_1 \cup F_2$, in fact it is clear that $F_1 \cup F_2 \subseteq V(\mathfrak{p}_0) \cap X = F$. On the other hand, if $\mathfrak{p} \in F$, then $\mathfrak{p}_0 \subseteq \mathfrak{p}$ and therefore $ab \in \mathfrak{p}$. But $\mathfrak{p}$ is prime, therefore $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. So $F = F_1 \cup F_2$. By irreducibility, we have $F = F_1$ or $F = F_2$, say $F = F_1$. But then $(\mathfrak{p}_0, a) \subseteq \mathfrak{p}_0$, that is $a \in \mathfrak{p}_0$ and hence $\mathfrak{p}_0$ is prime. □

Notation. Let R be a ring, $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ and let $\mathfrak{p} \in \mathrm{Spec}R$. Define

$$\mu_{\mathfrak{p}}(M) := \dim_{k(\mathfrak{p})} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = \mu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

where the last equality follows by Nakayama's lemma.

Lemma 70. *Let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$. The set*

$$F_t = \{\mathfrak{q} \in \mathrm{Spec}R \mid \mu_{\mathfrak{q}}(M) \geqslant t\} \subseteq \mathrm{Spec}R$$

is closed.

Proof. For $t \in \mathbb{N}^*$ consider the ideal

$$I_t = \sum_{m_{i_1},\ldots,m_{i_{t-1}} \in M} \left[(m_{i_1},\ldots,m_{i_{t-1}}) :_R M\right],$$

i.e. the sum of colons of M into any submodule of M generated by $t-1$ elements. Suppose $\mathfrak{p} \in \mathrm{Spec}R$ is such that $I_t \subseteq \mathfrak{p}$. Then $\mu_{\mathfrak{p}}(M) \geqslant t$, in fact if not there exist $t-1$ elements $m_1,\ldots,m_{t-1}$ such that $(m_1\ldots,m_{t-1})_{\mathfrak{p}} = M_{\mathfrak{p}}$, and since M is finitely generated this implies $(m_1,\ldots,m_{t-1}) :_R M \not\subseteq \mathfrak{p}$ (basically clearing denominators). On the other hand, if $I_t \not\subseteq \mathfrak{p}$, then there exist $m_1,\ldots,m_{t-1}$ such that $(m_1,\ldots,m_{t-1}) :_R M \not\subseteq \mathfrak{p}$, that is $M_{\mathfrak{p}} = (m_1,\ldots,m_{t-1})_{\mathfrak{p}}$, and therefore $\mu_{\mathfrak{p}}(M) \leqslant t-1$. This shows that I_t defines F_t. i.e. $F_t = V(I_t)$ is closed. □

Crucial Lemma 71. *Let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ and let $X \subseteq \mathrm{Spec}R$ be basic. Then there exists a finite set of primes $\Lambda \subseteq X$ such that if $\mathfrak{p} \in X \smallsetminus \Lambda$, there exists $\mathfrak{q} \subsetneq \mathfrak{p}$ such that*

$$\mu_{\mathfrak{q}}(M) = \mu_{\mathfrak{p}}(M).$$

Proof. By Lemma 70, for all $t \in \mathbb{N}^*$ F_t is a closed subset of $\mathrm{Spec}R$. By Proposition 69 $F_t \cap X$ is a finite union of irreducible closed sets, that is

$$F_t \cap X = \bigcup_{i \in \Lambda_t} V(\mathfrak{p}_{i,t}) \cap X,$$

for prime ideals $\mathfrak{p}_{i,t} \in F_t$. Define

$$\Lambda := \bigcup \{\mathfrak{p}_{i,t} \mid t = 0,\ldots,\mu(M), i \in \Lambda_t\} \subseteq X,$$

which is a finite set. If $\mathfrak{p} \in X \smallsetminus \Lambda$, set $t = \mu_{\mathfrak{p}}(M)$. Then $\mathfrak{p} \in F_t \cap X$, i.e. $p \in V(\mathfrak{p}_{i,t})$ for some $i \in \Lambda_t$. Then $\mathfrak{p}_{i,t} \subsetneq \mathfrak{p}$, and the containment is strict because $p \notin \Lambda$, while $\mathfrak{p}_{i,t} \in \Lambda$. But by definition $\mathfrak{p}_{i,t}$ has at least t generators, therefore

$$t = \mu_{\mathfrak{p}}(M) \geqslant \mu_{\mathfrak{p}_{i,t}}(M) \geqslant t,$$

where the second inequality follows from the fact that $\mathfrak{p}_{i,t} \subseteq \mathfrak{p}$, and therefore the minimal number of generators potentially decreases when further localizing at $\mathfrak{p}_{i,t}$. But then we have

$$\mu_{\mathfrak{p}}(M) = \mu_{\mathfrak{p}_{i,t}}(M).$$

□

Definition. Let R be a ring and let M be an R-module. Then $x \in M$ is a $\mathfrak{p}$-basic element if

$$\mu_{\mathfrak{p}}(M) > \mu_{\mathfrak{p}}(M/Rx),$$

which means that x is part of a minimal generating set of $M_{\mathfrak{p}}$. An element $x \in M$ is called X-basic if it is $\mathfrak{p}$-basic for all $\mathfrak{p} \in X$.

Theorem 72 (Eisenbud-Evans)**.** *Let R be a commutative ring with 1_R (not necessarily Noetherian). Let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ and let X be a basic set. Assume that*

(1) $(a, y) \in R \oplus M$ is X-basic.

(2) For all $\mathfrak{p} \in X$ we have $\mu_{\mathfrak{p}}(M) \geqslant 1 + \dim_X \mathfrak{p}$.

Then there exists $z \in M$ such that $y + az$ is X-basic.

To prove the theorem we need a few more auxiliary definitions and results.

Notation. In the assumptions of the theorem, for $\mathfrak{p} \in X$ and for $S \subseteq M$ set

$$\delta_{\mathfrak{p}}(S) := \mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/RS),$$

where RS denotes the R-submodule of M generated by the elements in S.

Remark 63. There is a short exact sequence

$$0 \longrightarrow RS \longrightarrow M \longrightarrow M/RS \longrightarrow 0.$$

Localizing at $\mathfrak{p}$ and then tensoring with $k(\mathfrak{p})$ gives

$$\frac{R_{\mathfrak{p}}S}{\mathfrak{p}R_{\mathfrak{p}}S} \xrightarrow{\alpha} \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}} \longrightarrow \frac{M_p}{R_{\mathfrak{p}}S + \mathfrak{p}M_{\mathfrak{p}}} \longrightarrow 0.$$

These are $k(\mathfrak{p})$-vector spaces and, if we denote by $k(\mathfrak{p})S = \mathrm{Im}(\alpha)$, we have

$$\delta_{\mathfrak{p}}(S) = \dim_{k(\mathfrak{p})} k(\mathfrak{p})S.$$

Remark 64. Let $X \subseteq \mathrm{Spec}R$ be basic and let $S \subseteq M$ be as above. Let $\Lambda \subseteq X$ be as in the Crucial Lemma 71. Then for all $\mathfrak{p} \in X \smallsetminus \Lambda$ there exists $q \in X$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\delta_{\mathfrak{q}}(S) \leqslant \delta_{\mathfrak{p}}(S)$.

Proof. Let $M' = M/RS$. By the Crucial Lemma 71 we know that except for finitely many primes $\mathfrak{p} \in X$ (the ones not in Λ) there exists $\mathfrak{q} \subsetneq \mathfrak{p}$ such that $\mu_{\mathfrak{p}}(M') = \mu_{\mathfrak{q}}(M')$. Also, $\mu_{\mathfrak{p}}(M) \geqslant \mu_{\mathfrak{q}}(M)$ always holds since it is a further localization. But then

$$\delta_{\mathfrak{q}}(S) = \mu_{\mathfrak{q}}(M) - \mu_{\mathfrak{q}}(M') \leqslant \mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M') = \delta_{\mathfrak{p}}(S).$$

□

Definition. Let X be basic and let $\mathfrak{p} \in X$. A subset $S \subseteq M$, where $S = \{x_1, \dots, x_n\}$ is called $\mathfrak{p}$-basic if

$$\delta_{\mathfrak{p}}(S) \geqslant \min\{n, 1 + \dim_X \mathfrak{p}\}.$$

If S is $\mathfrak{p}$-basic for all $\mathfrak{p} \in X$, then S is said to be X-basic.

Remark 65. Suppose $S = \{x\}$ and $\mathfrak{p} \in X$. Then S is $\mathfrak{p}$-basic if and only if x is $\mathfrak{p}$-basic, i.e. the two definitions coincide when S consists of one element.

Proof. If S is $\mathfrak{p}$-basic, then $\delta_{\mathfrak{p}}(S) \geqslant \min\{1, 1 + \dim_X \mathfrak{p}\} = 1$, then

$$\mu_{\mathfrak{p}}(M/Rx) < \mu_{\mathfrak{p}}(M),$$

i.e. x is $\mathfrak{p}$-basic. Conversely, if x is $\mathfrak{p}$-basic, we have $\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/Rx) \geqslant 1$, that is $\delta_{\mathfrak{p}}(S) \geqslant 1 = \min\{1, 1 + \dim_X \mathfrak{p}\}$, so that S is $\mathfrak{p}$-basic. Notice that we always have $\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/Rx) \leqslant 1$, therefore if $S = \{x\}$ is $\mathfrak{p}$-basic we necessarily have $\delta_{\mathfrak{p}}(S) = 1$. □

Remark 66. Suppose X is basic and M satisfies (2) in Theorem 72, that is $\mu_{\mathfrak{p}}(M) \geqslant 1 + \dim_X \mathfrak{p}$ for all $\mathfrak{p} \in X$. Let S be any (finite) set of generators of M. Then for all $\mathfrak{p} \in X$

$$\delta_{\mathfrak{p}}(S) = \mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(0) = \mu_{\mathfrak{p}}(M) \geqslant 1 + \dim_X \mathfrak{p} \geqslant \min\{|S|, 1 + \dim_X \mathfrak{p}\},$$

and therefore S is X-basic.

Main Lemma 73. *Let $S = \{x_1, \dots, x_n\} \subseteq M$ be X-basic. Assume that $(a, x_1) \in R \oplus M$ is X-basic. Then there exist $a_1, \dots, a_{n-1} \in R$ such that*

$$S' = \{x'_1, \dots, x'_{n-1}\} = \{x_1 + aa_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n\}$$

is X-basic.

Proof. We claim that for any choice of $a_1, \dots, a_{n-1}$, S' is $\mathfrak{p}$-basic for all but finitely many primes in X. In fact recall Remark 64, and let $\mathfrak{p} \in X \setminus \Lambda$. Notice that $R(S' \cup \{x_n\}) = RS$, and therefore

$$\delta_{\mathfrak{p}}(S') = \delta_{\mathfrak{p}}(S) \geqslant \min\{n, 1 + \dim_X \mathfrak{p}\},$$

because S is X-basics by assumption. But we also have

$$\mu_{\mathfrak{p}}(M/RS) = \mu_{\mathfrak{p}}(M/R(S' \cup \{x_n\})) \leqslant \mu_{\mathfrak{p}}(M/RS') - 1,$$

therefore $\delta_{\mathfrak{p}}(S') \geqslant \delta_{\mathfrak{p}}(S) - 1$. By Remark 64 there exists $q \subsetneq \mathfrak{p}$, $\mathfrak{q} \in X$ such that $\delta_{\mathfrak{q}}(S) \leqslant \delta_{\mathfrak{p}}(S)$. Putting things together:

$$\delta_{\mathfrak{p}}(S') \geqslant \delta_{\mathfrak{p}}(S) - 1 \geqslant \delta_{\mathfrak{q}}(S) - 1 \geqslant \min\{n, \dim_X \mathfrak{q}\} - 1 \geqslant$$

$$\geqslant \min\{n - 1, \dim_X \mathfrak{q}\} \geqslant \min\{n - 1, 1 + \dim_X \mathfrak{p}\},$$

where the last inequality follows from the fact that the containment $\mathfrak{q} \subsetneq \mathfrak{p}$ is strict. Now set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} = \Lambda$, which are the primes for which the claim does not hold. Choose $\mathfrak{p}_r$ minimal among the $\mathfrak{p}_i$'s in Λ. By induction on r we assume that we can choose $a_1, \dots, a_{n-1}$ such that S' is $\mathfrak{p}_i$-basic for all $i = 1, \dots, r-1$. By minimality we can now choose $c \in \mathfrak{p}_1 \cdot \dots \cdot \mathfrak{p}_{r-1} \smallsetminus \mathfrak{p}_r$. Set

$$\begin{cases} x_1'' = x_1' + acb_1x_n \\ x_2'' = x_2' + cb_2x_n \\ \dots\dots\dots \\ \dots\dots\dots \\ \dots\dots\dots \\ x_{n-1}'' = n_{n-1}' + cb_{n-1}x_n \end{cases}$$

for some $b_1, \dots, b_{n-1}$ to be determined. Set $S'' = \{x_1'', \dots, x_n''\}$. Fix $1 \leqslant i \leqslant r-1$ and set $M(\mathfrak{p}_i) = M \otimes k(\mathfrak{p}_i)$. Since $c \in \mathfrak{q}_i$ we have $\overline{x_j''} = \overline{x_j'}$ in $M(\mathfrak{p}_i)$, for all $j = 1, \dots, n-1$. Therefore $\delta_{\mathfrak{p}_i}(S') = \delta_{\mathfrak{p}_i}(S'')$, and so S'' is $\mathfrak{p}_i$-basic since S' is. Now we need to choose $b_1, \dots, b_{n-1}$ so that S'' is also $\mathfrak{p}_r$-basic. We distinguish three cases:

(a) $\overline{x_1'}, \dots, \overline{x_{n-1}'}$ are linearly independent in $M(\mathfrak{p}_r)$. Then we have a short exact sequence

$$0 \longrightarrow \sum_{i=1}^{n-1} k(\mathfrak{p}_r)\overline{x_i'} \longrightarrow M(\mathfrak{p}_r) \longrightarrow \frac{M(\mathfrak{p}_r)}{\sum_{i=1}^{n-1} k(\mathfrak{p}_r)\overline{x_i'}} = \frac{M}{RS'}(\mathfrak{p}_r) \longrightarrow 0.$$

Since they are linearly independent we get

$$\delta_{\mathfrak{p}_r}(S') = n-1 \geqslant \min\{n-1, 1+\dim_X \mathfrak{p}_r\},$$

so just take $b_1 = \ldots = b_{n-1} = 0$.

(b) $\overline{x_1'} = 0$. Since (a, x_1) is X-basic by assumption, we have $a \notin \mathfrak{p}_r$, otherwise $\overline{(a, x_1)} = 0$ in $(R \oplus M) \otimes k(\mathfrak{p}_r)$, and it cannot be basic. Set $b_1 = 1$, $b_2 = \ldots = b_{n-1} = 0$. The image of S'' in $M(\mathfrak{p}_r)$ is then

$$k(\mathfrak{p}_r)S = k(\mathfrak{p}_r)\{\overline{acx_n}, \overline{x_2'}, \dots, \overline{x_{n-1}'}\}.$$

By choice of c and by what we just observed we have that $\overline{ac}$ is a unit in $k(\mathfrak{p}_r)$, hence

$$k(\mathfrak{p}_r)S'' = k(\mathfrak{p}_r)\{\overline{x_2'}, \dots, \overline{x_{n-1}'}, \overline{x_n}\}.$$

But $x_i' = x_i + cb_ix_n$, therefore

$$k(\mathfrak{p}_r)S'' = k(\mathfrak{p}_r)\{\overline{x_2}, \dots, \overline{x_{n-1}}, \overline{x_n}\}.$$

Finally, by definition of x_1' we have $0 = \overline{x_1'} = \overline{x_1} + \lambda x_n$ for some λ, and therefore $\overline{x_1} = \overline{\lambda x_n}$ in $k(\mathfrak{p}_r)$. Therefore

$$k(\mathfrak{p}_r)S'' = k(\mathfrak{p}_r)\{\overline{x_1}, \dots, \overline{x_n}\} = k(\mathfrak{p}_r)S,$$

which gives

$$\delta_{\mathfrak{p}_r}(S'') = \delta_{\mathfrak{p}_r}(S) \geqslant \min\{n, 1 + \dim_X \mathfrak{p}_r\} \geqslant \min\{n-1, 1 + \dim_X \mathfrak{p}_r\},$$

that is S'' is $\mathfrak{p}_r$ basic.

(c) $\overline{x_1'} \neq 0$ and $\overline{x_1'}, \ldots, \overline{x_{n-1}'}$ are linearly dependent in $k(\mathfrak{p}_r)$. There exists $1 \leqslant i \leqslant n-1$ such that $\overline{x_i'}$ is not zero and can be expressed in terms of the others. Set $b_i = 1$ and $b_j = 0$ for all $i \neq j$. Then

$$k(\mathfrak{p}_r)S'' = k(\mathfrak{p}_r)\{\overline{x_1'}, \ldots, \overline{x_i'} + \overline{cx_n}, \ldots, \overline{x_{n-1}'}\} =$$

$$= k(\mathfrak{p}_r)\{\overline{x_1'}, \ldots, \widehat{\overline{x_i}}, \ldots, \overline{x_{n-1}'}, \ldots, \overline{cx_n}\}.$$

because $\overline{x_i'}$ can be expressed in terms of the others. Now, $\bar{c}$ is a unit in $k(\mathfrak{p}_r)$, therefore

$$k(\mathfrak{p}_r)S'' = k(\mathfrak{p}_r)\{\overline{x_1'}, \ldots, \widehat{\overline{x_i}}, \ldots, \overline{x_{n-1}'}, \overline{x_n}\} = k(\mathfrak{p}_r)\{\overline{x_1}, \ldots, \widehat{\overline{x_i}}, \ldots, \overline{x_n}\}.$$

Finally, since

$$\overline{x_i'} = \overline{x_i} + \lambda \overline{x_n} \in k(\mathfrak{p}_r)\{\overline{x_1'}, \ldots, \widehat{\overline{x_i'}}, \ldots, \overline{x_{n-1}'}, \overline{x_n}\} = k(\mathfrak{p}_r)\{\overline{x_1}, \ldots, \widehat{\overline{x_i}}, \ldots, \overline{x_n}\}$$

we also get $x_i \in k(\mathfrak{p}_r)\{\overline{x_1}, \ldots, \widehat{\overline{x_i}}, \ldots, \overline{x_n}\}$, that is

$$k(\mathfrak{p}_r)S'' = k(\mathfrak{p}_r)\{\overline{x_1}, \ldots, \overline{x_i}, \ldots, \overline{x_n}\} = k(\mathfrak{p}_r)S$$

and therefore S'' is $\mathfrak{p}_r$-basic again.

□

We are now ready to prove Eisenbud-Evans' theorem 72.

Proof of Theorem 72. Let $S = \{y, x_2, \ldots, x_n\}$ be a generating set of M. Set $x_1 := y$. By condition (2), that is $\mu_{\mathfrak{p}}(M) \geqslant 1 + \dim_X \mathfrak{p}$ for all $\mathfrak{p} \in X$ and by Remark 66 we have that x_1 is X-basic. By the Main Lemma 73 we can find $a_1, \ldots, a_{n-1}$ such that

$$S' = \{x_1 + aa_1x_n, x_2', \ldots, x_{n-1}'\}$$

is X-basic. Notice that the cardinality of S' is one less than the cardinality of S. Repeat the process until the cardinality of the set, say S'", is one. But then $S'' = \{x_1 + az\} = \{y + az\}$ for some $z \in M$, and S'' is X-basic. But by Remark 65 this exactly means that $y + az$ is X-basic. □

Corollary 74 (of Theorem 72)**.** *Let R be a commutative ring with 1_R, let X be basic and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ be such that $\mu_{\mathfrak{p}}(M) \geqslant 1 + \dim_X \mathfrak{p}$ for all $\mathfrak{p} \in X$ (i.e. it satisfies just (2) in Theorem 72). Then there exists $z \in M$ which is X-basic.*

Proof. Set $a = 1$, $y = 0$. Then $(1, 0)$ is always X-basic in $R \oplus M$, and therefore by Theorem 72 there exists $z \in M$ such that $z = y + az$ is X-basic. $\square$

Corollary 75 (of Theorem 72)**.** *Let R be a Noetherian ring of dimension d and let $I \subseteq R$ be an ideal. Then there exist $d+1$ elements $a_1, \dots, a_{d+1}$ such that*

$$I \subseteq \sqrt{(a_1, \dots, a_{d+1})}.$$

Proof. Let $M = \oplus^d I$, and take $X = \mathrm{Supp}M$, which is closed in $\mathrm{Spec}R$ (because M is finitely generated) and hence basic. Note that for $\mathfrak{p} \in X$

$$\mu_{\mathfrak{p}}(M) \geqslant (d+1)\mu_{\mathfrak{p}}(I) \geqslant d+1 \geqslant 1 + \dim_X \mathfrak{p}.$$

The first inequality follows from Nakayama's Lemma, while the second is because $\mathfrak{p} \in X = \mathrm{Supp}M$. By Corollary 74 there exists $z = (a_1, \dots, a_{d+1}) \in M$ which is X-basic.

Claim. $I \subseteq \sqrt{(a_1, \dots, a_{d+1})}$.

In fact if not there exists $\mathfrak{p} \supseteq (a_1, \dots, a_{d+1})$ such that $\mathfrak{p} \not\supseteq I$. But then

$$M_{\mathfrak{p}} = \oplus^d I_{\mathfrak{p}} = R_{\mathfrak{p}}^{d+1},$$

i.e. $(a_1, \dots, a_{d+1}) \in \mathfrak{p}R_{\mathfrak{p}}^{d+1}$. By Nakayama's Lemma this contradicts the fact that $z = (a_1, \dots, a_{d+1})$ is basic. $\square$

2 Basic elements and Projective modules

Question. *What does it mean for any element $z \in P$ to be basic at $\mathfrak{q} \in \mathrm{Spec}R$ if P is a finitely generated projective module?*

Discussion. Assume that $P = R^t$ is free, and that $z = (z_1, \dots, z_t) \in P$. Then z is $\mathfrak{q}$-basic if and only if it is a minimal generator for $R\mathfrak{p}^t$, i,e, $z \notin \mathfrak{q}R_{\mathfrak{q}}^t$. But this is equivalent to say that $(z_1, \dots, z_t) \not\subseteq \mathfrak{q}$ (where here we mean the ideal generated by $z_1, \dots, z_t$ in R). This means that $(z_1, \dots, z_t)$ is a unimodular row in $R_{\mathfrak{q}}^t$, and therefore

$$R_{\mathfrak{q}}^t \simeq zR \oplus R_{\mathfrak{q}}^{t-1}.$$

If P is any projective module, locally we have $P_{\mathfrak{q}} \simeq R_{\mathfrak{q}}^t$ for some t. Therefore $z \in P$ is $\mathfrak{q}$-basic if and only if

$$P_{\mathfrak{q}} \simeq R_{\mathfrak{q}}^t \simeq zR_{\mathfrak{q}} \oplus Q_{\mathfrak{q}}$$

for some $Q \in \mathrm{Mod}^{\mathrm{fg}}(R)$.

Question. *Given $P \in \mathrm{Mod}^{\mathrm{fg}}(R)$ a projective module and given $z \in P$, what is*

$$\mathcal{U}_z := \{\mathfrak{q} \in \mathrm{Spec}R \mid z \textit{ is } \mathfrak{q} - \textit{basic}\}?$$

Discussion. First suppose $P \simeq R^t$ is free. Then by the above discussion it is easy to see that for $z = (z_1, \dots, z_t) \in R^t$ we have

$$\mathcal{U}_z = \mathrm{Spec}R \setminus V((z_1, \dots, z_t)).$$

For any $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ define $M^*(z) := \{f(z) \mid f \in M^*\}$. If P is just projective and $z \in P$, then we get

$$\mathcal{U}_z = \mathrm{Spec}R \setminus V(P^*(z)),$$

since we can localize and easily reduce to the free case.

Corollary 76 (Serre's Theorem)**.** *Let R be a Noetherian commutative ring with 1_R. Let $X = j\text{-Spec}R$ and $d = \dim(j\text{-Spec}R)$. Let $P \in fgR$ be a projective module and assume $\mathrm{rank}P_{\mathfrak{m}} > d$ for all $\mathfrak{m} \in \max\mathrm{Spec}R$. Then $P \simeq R \oplus Q$.*

Proof. First notice that it is stated as a Corollary because it will follow from Corollary 74 of Theorem 72. Notice that for all $\mathfrak{q} \in X = j\text{-Spec}R$ we have (for $\mathfrak{q} \subseteq \mathfrak{m} \in \max\mathrm{Spec}R$)

$$\mu_{\mathfrak{q}}(P) = \mathrm{rank}P_{\mathfrak{q}} = \mathrm{rank}P_{\mathfrak{m}} \geqslant d + 1 \geqslant \dim_X \mathfrak{q} + 1.$$

Therefore there exists a X-basic element $z \in P$, and then $P^*(z) = R$, i.e. $P \simeq Rz \oplus Q$. $\square$

Definition. With the same notation as in Chapter 2, we say that $n \in \mathbb{N}$ defines a stable range for $GL(R)$ if whenever $r > n$ and $(a_1, \dots, a_r)$ is unimodular, then there exist $b_1, \dots, b_{r-1} \in R$ such that $(a_1 + a_r b_1, a_2 + a_r b_2, \dots, a_{r-1} + a_r b_{r-1})$ is unimodular.

We need the following theorem. We are going to prove just the first part.

Theorem 77. *If n defines a stable range of $GL(R)$, then*

(1) $\dfrac{GL_m(R)}{E_m(R)} \to \dfrac{GL(R)}{E(R)}$ *is onto for all $m \geqslant n$.*

(2) $E_r(R) \trianglelefteq GL_r(R)$ *for all r.*

(3) $GL_r(R)/E_r(R)$ *is abelian for $r \geqslant 2n$.*

Lemma 78. *Let n define a stable range for $GL(R)$ and let $r > n$. If $(a_1, \dots, a_r)$ is unimodular, then there exists $A \in E_r(R)$ such that*

$$((a_1, \dots, a_r)A)^t = (1, 0, \dots, 0)^t.$$

Sketch. By adding multiples of the last row to the first $r-1$ rows we can assume that $a_1, \dots, a_{r-1}$ is unimodular. Then, since it is unimodular, one can add multiples of the first $r-1$ rows to get $(a_1, \dots, a_{r-1}, 1)$. Now, adding multiples of 1 we can clearly get rid of $a_1, \dots, a_{r-1}$, and finally get to $(0, \dots, 0, 1)$. $\square$

Remark 67. With $r > n$ as above we have that $GL_r(R)E_{r+1}(R) = GL_{r+1}(R)$.

Proof. Clearly $GL_r(R)E_{r+1}(R) \subseteq GL_{r+1}(R)$. If $A \in GL_{r+1}(R)$, then the last row is unimodular, so there exists $E \in E_{r+1}(R)$ such that

$$AE = \begin{bmatrix} A' & f \\ 0 & 1 \end{bmatrix},$$

for $A' \in GL_r(R)$. But notice that

$$\begin{bmatrix} I & * \\ 0 & 1 \end{bmatrix} \in E_{r+1}(R),$$

and hence

$$AE' = AE \begin{bmatrix} I & (-A')^{-1}f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A' & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & (-A')^{-1}f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix},$$

with $E' \in E_{r+1}(R)$. But this means

$$A = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix} (E')^{-1}.$$

This also proves (1) in Theorem 77. $\square$

Theorem 79 (Bass' Stable Range Theorem)**.** *Let R be a commutative Noetherian ring with 1_R. Then $n = \dim(j\text{-Spec}R)+1$ defines a stable range for $GL(R)$.*

Proof. Let $r > n = \dim(j\text{-Spec}R)+1$, and let $(a_1, \ldots, a_r)$ be unimodular in R^r. Let $M = R^{r-1}$ be the first $r-1$ copies of R, so that $y = (a_1, \ldots, a_{r-1}) \in M$. Then $(y, a) \in M \oplus R$, with $a := a_r$. Since (a, y) is unimodular, it is in particular basic. Also, for all $\mathfrak{q} \in j\text{-Spec}R$ we have $M_{\mathfrak{q}} \simeq R_{\mathfrak{q}}^{r-1}$, therefore

$$\mu_{\mathfrak{q}}(M) = r - 1 > \dim(j\text{-Spec}R),$$

and hence $\mu_{\mathfrak{q}}(M) \geqslant \dim(j\text{-Spec}R) + 1 \geqslant \dim_{j\text{-Spec}R} \mathfrak{q} + 1$, i.e. (2) of Theorem 72 holds as well. Then there exists $z \in M$ which is $j\text{-Spec}R$ basic, i.e. there is $z = (b_1, \ldots, b_{r-1}) \in M$ such that

$$y + az = (a_1 + a_r b_1, \ldots, a_{r-1} + a_r b_{r-1})$$

is $j\text{-Spec}R$ basic, that is $(a_1 + a_r b_1, \ldots, a_{r-1} + a_r b_{r-1})$ is unimodular. $\square$

Theorem 80 (Bourbaki's Theorem)**.** *Let R be an integrally closed Noetherian domain, and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ be a torsion free module of rank r. Then there exists an ideal $I \subseteq R$ and an exact sequence*

$$0 \longrightarrow R^{r-1} \longrightarrow M \longrightarrow I \longrightarrow 0.$$

Notice that in particular this implies that $cl(M) = cl(I)$.

Proof. Let us induct on $r = \mathrm{rank} M$.

- If $r = 1$ then $M \simeq I$ is an ideal.
- If $r > 1$, then $\mu_{\mathfrak{q}}(M) \geqslant \mathrm{rank} M_{\mathfrak{q}} \geqslant 2$ for all $\mathfrak{q} \in \mathrm{Spec} R$. Apply Corollary 74 to $X = X^1 = \{\mathfrak{q} \in \mathrm{Spec} R \mid \mathrm{ht}\mathfrak{q} \geqslant 1\}$. Notice that condition (2) of Theorem 72 is satisfied, therefore there exists $x = x_1 \in M$ which is X^1-basic, with $R \simeq Rx \subseteq M$ (because M is torsion free).

  Claim. $N := M/R$ is torsion free.

  In fact let $0 \neq z \in R$, we need to prove that z is a nonzero divisor in N. Consider the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & R \arrow[r] \arrow[d, "\cdot z"'] & M \arrow[r] \arrow[d, "\cdot z"'] & N \arrow[r] \arrow[d, dashed, "\mu_z = \cdot z"] & 0 \\
0 \arrow[r] & R \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

  where the first two vertical maps are inceptive because R is a domain, and M is torsion free. Also, μ_z is just the map induced by the diagram, and it is again multiplication by z. By the Snake Lemma we get an inclusion

$$0 \longrightarrow \ker \mu_z \longrightarrow \mathrm{coker}(R \xrightarrow{\cdot z} R) = R/Rz.$$

  It is enough to show then that $\mathrm{Ass}(\ker \mu_z) = \varnothing$. Let $\mathfrak{p} \in \mathrm{Ass}(\ker \mu_z)$, then we have inclusions

$$R/\mathfrak{p} \hookrightarrow \ker \mu_z \hookrightarrow R/Rz,$$

  and therefore $\mathfrak{p} \in \mathrm{Ass}(R/Rz)$. But R is an integrally closed domain, and then z is a nonzero divisor in R. Therefore $\mathrm{ht}\mathfrak{p} = 1$. But localizing at $\mathfrak{p}$ we then have that $R_{\mathfrak{p}}$ is a DVR, and $M_{\mathfrak{p}}$ is torsion free over $R_{\mathfrak{p}}$. However, torsion free modules over DVR are free, that is $M_{\mathfrak{p}}$ is free. Recall that $x = x_1$ is X^1-basic, and therefore it is $\mathfrak{p}$-basic. This means that

$$M_{\mathfrak{p}} \simeq xR_{\mathfrak{p}} \oplus N_{\mathfrak{p}},$$

  since x is basic (i.e. unimodular) in a free module. This implies in particular that $N_{\mathfrak{p}}$ is torsion free, and therefore $(\ker \mu_z)_{\mathfrak{p}} = 0$. Since this happens for all $\mathfrak{p} \in \mathrm{Ass}(\ker \mu_z)$ we must have $\ker \mu_z = 0$, that is N is torsion free.

Now apply the inductive hypothesis to N (since $\mathrm{rank} N = r - 1 < \mathrm{rank} M$) to get an ideal $I \subseteq R$ and a short exact sequence

$$0 \longrightarrow R^{r-2} \longrightarrow N \longrightarrow I \longrightarrow 0.$$

Since $N \simeq M/R$ we finally get

$$0 \longrightarrow R^{r-1} \longrightarrow M \longrightarrow I \longrightarrow 0.$$

∎

Remark 68. Recall that, given X basic and $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$, a submodule $M' \subseteq M$, $M' = R\{m_1, \dots, m_n\}$, is called X-basic if for all $\mathfrak{p} \in X$

$$\delta_{\mathfrak{p}}(M)' = \mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/M') \geqslant \min\{n, 1 + \dim_X \mathfrak{p}\}.$$

It follows from Theorem 72 that if $M' \subseteq M$ is X-basic, then there exists $z \in M'$ such that z is X-basic in M.

Theorem 81 (Bass' Cancellation Theorem)**.** *Let $d = \dim X$, where $X = j$-$\mathrm{Spec}R$. Let $P \in \mathrm{Mod}^{\mathrm{fg}}(R)$ be a projective module such that $\mathrm{rank}P_{\mathfrak{q}} \geqslant d+1$ for all $\mathfrak{q} \in X$ (which is equivalent to the same condition for all $\mathfrak{q} \in \mathrm{Spec}R$). Let $Q \in \mathrm{Mod}^{\mathrm{fg}}(R)$ be another projective module, and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ be any other module such that*

$$Q \oplus M \simeq Q \oplus P.$$

Then $M \simeq P$.

Proof. Notice that M is automatically projective to start with, since we are assuming that $Q \oplus M \simeq Q \oplus P$, and both Q and P are projective. Then we can choose Q' a finitely generated projective R-module such that $Q \oplus Q' \simeq R^n$, i.e. we can assume that

$$R^n \oplus M \simeq R^n \oplus P.$$

Also, by induction it is enough to show that $M \simeq P$ whenever $R \oplus M \simeq R \oplus P$. Set $\alpha : R \oplus M \to R \oplus P$ the isomorphism, and notice that if $\alpha((1,0)) = (1,0)$, then clearly we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "\mathrm{id}_R"', "\simeq"] & R \oplus M \arrow[r] \arrow[d, "\simeq"', "h"] & M \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & R \arrow[r] & R \oplus P \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

so that $M \simeq P$. If this is not the case, let $\alpha((1,0)) = (a, x_1)$. The goal is to show that

$$\begin{tikzcd}
R \oplus M \arrow[r, "\alpha"] & R \oplus P \arrow[r, "\beta"] & R \oplus P \arrow[r, "\gamma"] & R \oplus P \arrow[r, "\eta"] & R \oplus P \\
(1,0) \arrow[r, mapsto] & (a, x_1) \arrow[rr, mapsto] & & (1,*) \arrow[r, mapsto] & (1,0)
\end{tikzcd}$$

are all isomorphism, with β, γ and η to be defined, in order to repeat the argument above. Notice that, since $(1,0)$ is basic in $R \oplus M$ and α is an isomorphism, $\alpha((1,0)) = (a, x_1)$ is basic in $R \oplus P$. Write $P = R\{x_1, \dots, x_n\}$, and notice that for all $\mathfrak{p} \in X$ we have

$$\mathrm{rank}P_{\mathfrak{p}} = \mu_{\mathfrak{p}}(P) \geqslant 1 + \dim_X \mathfrak{p}$$

by assumption. By Theorem 72 there exists $z = x_1 + ax \in P$ which is X-basic. Write $x = \sum_{i=1}^{n} r_i x_i$. Define

$$\begin{aligned} f : R &\longrightarrow P \\ 1 &\longmapsto x \end{aligned}$$

Now, the map $\beta : R \oplus P \to R \oplus P$ defined by the matrix

$$\begin{bmatrix} 1_R & 0 \\ f & 1_P \end{bmatrix}$$

is an isomorphism since $\det \beta = 1$. Also,

$$\beta(\alpha((1,0))) = \beta((a, x_1)) = \left(\begin{bmatrix} 1_R & 0 \\ f & 1_P \end{bmatrix} \begin{pmatrix} a \\ x_1 \end{pmatrix} \right)^t = (a, f(a) + x_1) = (a, z).$$

Recall that, since z is basic, we have $P \simeq Rz \oplus P'$, and hence there is a splitting map $\varphi : P \to R$ such that $\varphi(z) = 1 - a$. Define $\gamma : R \oplus P \to R \oplus P$ via the matrix

$$\begin{bmatrix} 1_R & \varphi \\ 0 & 1_P \end{bmatrix},$$

and notice that it is again an isomorphism since $\det \gamma = 1$. Also notice that

$$\gamma(\beta(\alpha((0,1)))) = \gamma((a, z)) = \left(\begin{bmatrix} 1_R & \varphi \\ 0 & 1_P \end{bmatrix} \begin{pmatrix} a \\ z \end{pmatrix} \right)^t = (a + \varphi(z), z) = (1, z).$$

Finally, let

$$\begin{aligned} g : R &\longrightarrow P \\ 1 &\longmapsto z \end{aligned}$$

and define $\eta : R \oplus P \to R \oplus P$ via the matrix

$$\begin{bmatrix} 1_R & 0 \\ -g & 1_P \end{bmatrix},$$

which has determinant one, and hence it is again an isomorphism. We have

$$\eta(\gamma(\beta(\alpha((1,0))))) = \eta((1, z)) = \left(\begin{bmatrix} 1_R & 0 \\ -g & 1_P \end{bmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} \right)^t = (1, 0).$$

This proves the theorem. $\square$

Theorem 82 (Forester-Swan)**.** *Let R be a Noetherian ring, and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$. Set $X = j\text{-}\mathrm{Spec}R \cap \mathrm{Supp}M$, which is basic since $\mathrm{Supp}M$ is closed in $\mathrm{Spec}R$ and $j\text{-}\mathrm{Spec}R$ is basic. Then*

$$\mu(M) \leqslant \sup_{\mathfrak{p} \in X} \{\dim_X \mathfrak{p} + \mu_{\mathfrak{p}}(M)\}.$$

Proof. Set $n := \mu(M)$ and $t := \sup_{\mathfrak{p}\in X}\{\dim_X \mathfrak{p} + \mu_{\mathfrak{p}}(M)\}$. We want to show that $n \leqslant t$. Since we can pass to $R/ann(M)$ we can assume without loss of generality that $X = j\text{-Spec}R$. By way of contradiction assume that $n > t$, which in particular means that $t < \infty$ since M is finitely generated. Since M is minimally generated by n elements there exists a short exact sequence

$$0 \longrightarrow M' \longrightarrow R^n \longrightarrow M \longrightarrow 0,$$

and tensoring with $k(\mathfrak{p})$ for $\mathfrak{p} \in X$ we get

$$M' \otimes k(\mathfrak{p}) \longrightarrow k(\mathfrak{p})^n \longrightarrow M \otimes k(\mathfrak{p}) \longrightarrow 0.$$

Counting dimensions:

$$\dim_{k(\mathfrak{p})}\left(\mathrm{Im}\left(M' \otimes k(\mathfrak{p}) \to k(\mathfrak{p})^n\right)\right) = n - \mu_{\mathfrak{p}}(M) > t - \mu_{\mathfrak{p}}(M) \geqslant \dim_X \mathfrak{p}.$$

Therefore

$$\dim_{k(\mathfrak{p})}\left(\mathrm{Im}\left(M' \otimes k(\mathfrak{p}) \to k(\mathfrak{p})^n\right)\right) \geqslant 1 + \dim_X \mathfrak{p}$$

for all $\mathfrak{p} \in X$, and hence M' is X-basic. But then, by Corollary 74 there exists $z \in M'$ which is basic in R^n, i.e. it is unimodular (since R^n is free). This means

$$R^n \simeq Rz \oplus P.$$

But $\mathrm{rank}P = n - 1 \geqslant t$, therefore we have $P \simeq R^{n-1}$ by Bass cancellation Thorem 81. Since $z \in M'$, we still have a surjection

$$R^n/Rz \longrightarrow M \longrightarrow 0,$$

and by what we have just shown we have $R^n/Rz \simeq P \simeq R^{n-1}$, which gives a surjection

$$R^{n-1} \longrightarrow M \longrightarrow 0,$$

contradicting the minimality of $n = \mu(M)$. □

Corollary 83. *In a Dedekind domain D every ideal is minimally generated by at most two elements.*

Proof. Any ideal $I \subseteq D$ is projective, since it is locally principal, i.e. $\mu_{\mathfrak{p}}(I) = 1$ for all $\mathfrak{p} \in \mathrm{Spec}D$. If $\mathfrak{p}$ is maximal, then $\dim_X \mathfrak{p} = 0$, therefore $\dim_X \mathfrak{p} + \mu_{\mathfrak{p}}(I) = 1$ in this case. If $\mathfrak{p} = 0$, then $\dim_X 0 + \mu_{(0)}(I) \leqslant 1 + 1 = 2$. Therefore

$$\mu(I) \leqslant \sup_{\mathfrak{p}\in\mathrm{Spec}D}\{\dim_X \mathfrak{p} + \mu_{\mathfrak{p}}(()\, I)\} = 2.$$

□

Remark 69. Notice that, for example when D is semi-local, $(0) \notin X = j\text{-Spec}D$, so in this case if $0 \neq I \subseteq D$ is an ideal we have

$$\mu(I) = \sup_{\mathfrak{p}\in X}\{\dim_X \mathfrak{p} + \mu_{\mathfrak{p}}(I)\} = 0 + 1 = 1,$$

so ideals are principally generated, and D is a PID.

Example 39. Let $R = k[x_1, \ldots, x_n]$ and let $M = \mathfrak{m}$ be a maximal ideal. By the Nullstellensatz we know that j-Spec$R$ = Spec$R$. If $\mathfrak{p} \neq \mathfrak{m}$ we get

$$\dim_X \mathfrak{p} + \mu_{\mathfrak{p}}(\mathfrak{m}) = \dim(R/\mathfrak{p}) + \mu_{\mathfrak{p}}(R)_{\mathfrak{p}} = \dim(R/\mathfrak{p}) + 1,$$

where $\dim(R/\mathfrak{p})$ is the Krull dimension. If $\mathfrak{p} = \mathfrak{m}$ we get

$$\dim_X \mathfrak{m} + \mu_{\mathfrak{m}} \mathfrak{m} = 0 + n,$$

therefore

$$\sup_{\mathfrak{p} \in \mathrm{Spec} R} \{\dim(R/\mathfrak{p}) + 1, n\} = \dim(R/0) + 1 = n + 1.$$

Hence by Theorem 82 we know that $\mu(\mathfrak{m}) \leqslant n + 1$. Notice that this is off by one, since we know that $\mu(\mathfrak{m}) = n$.

Conjecture 1 (Proved)**.** *For $R = k[x_1, \ldots, x_n]$ one can exclude the prime $\mathfrak{p} = 0$, so that in the example above we get a tight upper bound.*

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